

# CAUCHY PROBLEMS FOR LORENTZIAN MANIFOLDS WITH SPECIAL HOLONOMY

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**ABSTRACT.** On a Lorentzian manifold the existence of a parallel null vector field implies certain constraint conditions on the induced Riemannian geometry of a space-like hypersurface. We will derive these constraint conditions and, conversely, show that every real analytic Riemannian manifold satisfying the constraint conditions can be extended to a Lorentzian manifold with a parallel null vector field. Similarly, every parallel null spinor on a Lorentzian manifold induces an imaginary generalised Killing spinor on a space-like hypersurface. Then, based on the fact that a parallel spinor field induces a parallel vector field, we can apply the first result to prove: every real analytic Riemannian manifold carrying a real analytic, imaginary generalised Killing spinor can be extended to a Lorentzian manifold with a parallel null spinor. Finally, we give examples of geodesically complete Riemannian manifolds satisfying the constraint conditions.

## 1. BACKGROUND AND MAIN RESULTS

This paper is a contribution to the research programme of studying global and causal properties of Lorentzian manifolds with special holonomy. A Lorentzian manifold has *special holonomy* if the connected component of its holonomy group is reduced from the full group  $\mathbf{SO}^0(1, n)$ , but still acts indecomposably, i.e., without non-degenerate invariant subspaces. In this situation the Lorentzian manifold admits a bundle of tangent null lines that is invariant under parallel transport. The possible special Lorentzian holonomy groups were classified in [7] and [14], all of them can be realised by local metrics [12], but many questions about the consequences of special holonomy for global and causal properties of the manifold are still open. A special case of this situation is when the parallel null line bundle is spanned by a *parallel null vector field*. This is the case we will study in this paper. It is motivated by the question which Lorentzian manifolds admit a *parallel spinor field*, which in turn draws its motivation from mathematical physics. Since a parallel spinor is invariant under the spin representation of the holonomy group, indecomposable Lorentzian manifolds with parallel spinors have special holonomy. However, since  $\mathbf{SO}^0(1, n)$  has no proper *irreducible* subgroups, the situation is very different from the Riemannian case, where we have several *irreducible* holonomy groups that admit an invariant spinor. In fact, a spinor field  $\phi$  on any Lorentzian manifold  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  induces a causal vector field  $V_\phi$ , its *Dirac current*, which is defined by

$$\overline{\mathbf{g}}(X, V_\phi) = -\langle X \cdot \phi, \phi \rangle,$$

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2010 *Mathematics Subject Classification.* Primary 53C50, 53C27; Secondary 53C44, 35A10, 83C05.

*Key words and phrases.* Lorentzian manifolds, special holonomy, parallel spinors, parallel null vectors, generalised Killing spinors, Cauchy problem.

The authors acknowledge support from the Australian Research Council through the grants FT110100429 and DP120104582 and from the Collaborative Research Centre 647 “Space-Time-Matter” of the German Research Foundation.

for all  $X \in T\overline{\mathcal{M}}$ . If the spinor  $\phi$  is parallel,  $V_\phi$  is a parallel vector field and thus reduces the holonomy to its stabiliser. Moreover, if we assume that the manifold is indecomposable,  $V_\phi$  must be *null*, by which we mean  $\overline{\mathbf{g}}(V_\phi, V_\phi) = 0$  and  $V_\phi \neq 0$ .

A fundamental problem in the programme mentioned at the beginning is the question: What is the intersection of the class of globally hyperbolic Lorentzian manifolds with the class of Lorentzian manifolds with special holonomy? A Lorentzian manifold is *globally hyperbolic* if it admits a Cauchy hypersurface, i.e., a space-like hypersurface that is met by every inextendible timelike curve exactly once. It is known from work of Bernal and Sánchez [8] that a globally hyperbolic Lorentzian manifold  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  is of the form  $\overline{\mathcal{M}} = \mathbb{R} \times \mathcal{M}$  with the metric

$$\overline{\mathbf{g}} = -\lambda dt^2 + \mathbf{g}_t, \quad (1)$$

where  $\mathbf{g}_t$  is a  $t$ -dependent family of Riemannian metrics on  $\mathcal{M}$  and  $\lambda = \lambda(t, x)$  is a smooth function on  $\overline{\mathcal{M}}$ , the so-called *lapse function*. Hence, the first step in order to understand globally hyperbolic manifolds with special holonomy, and more specifically, with parallel null vector field or parallel null spinor field (for which  $V_\phi$  is null) is to investigate the following questions:

- (A) *Constraint conditions*: What are the *constraint conditions* that are imposed on the space-like hypersurface  $\mathcal{M}$  in the manifold  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  by the existence of (i) *parallel null vector field*, or (ii) *parallel null spinor field*?
- (B) *Cauchy problem*: Can we extend a given a Riemannian manifold satisfying these constraint conditions to a Lorentzian manifold with metric as in (1) with a (i) *parallel null vector field*, or (ii) *parallel null spinor field*?

By evaluating the Gauß-Codazzi equations we find the answer to question A(i): If  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  admits a parallel null vector field  $V$  then there is a vector field  $U = -\text{pr}_{T\mathcal{M}}(V)$  such that

$$\nabla^{\mathbf{g}}U + uW = 0, \quad (2)$$

with  $u^2 = \mathbf{g}(U, U)$  and in which  $W := -\overline{\nabla}T$  is the Weingarten operator of  $\mathcal{M} \subset (\overline{\mathcal{M}}, \overline{\mathbf{g}})$ . Hence question B(i) can be stated more precisely as: can a Riemannian manifold  $(\mathcal{M}, \mathbf{g})$  that is given together with a vector field  $U$  and a  $\mathbf{g}$ -symmetric endomorphism field  $W$  satisfying the constraint equation (2) be extended to Lorentzian manifold  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  as in (1) with a parallel null vector field  $V$  that projects to  $U$ . In Sections 3 and 4 we will derive a PDE system in the form of *evolution equations* and which is equivalent to the existence of a parallel null vector field  $V$  for the metric in (1). It is of the form

$$\partial_t^2 \mathcal{V} = F(\mathcal{V}, \partial_i \mathcal{V}, \partial_t \mathcal{V}, \partial_i \partial_j \mathcal{V}, \partial_i \partial_t \mathcal{V}) \quad (3)$$

with  $\mathcal{V}(t, x^i) = (\mathbf{g}(t, x^i), U(t, x^i), u(t, x^i))$  a triple of symmetric bilinear forms, vectors and functions depending on  $t$  and  $x^i$  (see Theorems 3.1 and 4.1 for a detailed statement). Thus we can apply the Cauchy-Kowalevski Theorem to (3), provided that the initial data and the lapse function are analytic, and obtain:

**Theorem 1.** *Let  $(\mathcal{M}, \mathbf{g}, W, U)$  be an analytic Riemannian manifold together with a field of  $\mathbf{g}$ -symmetric, analytic endomorphisms  $W$ , and an analytic vector field  $U$  satisfying the constraint equation (2). Then, for any analytic function  $\lambda$  on  $\mathbb{R} \times \mathcal{M}$  there exists an open neighbourhood  $\overline{\mathcal{U}}$  of  $\{0\} \times \mathcal{M}$  in  $\mathbb{R} \times \mathcal{M}$  and an unique analytic Lorentzian metric*

$$\overline{\mathbf{g}} = -\lambda^2 dt^2 + \mathbf{g}_t$$

*on  $\overline{\mathcal{U}}$  which admits an analytic, parallel null vector field  $V = \frac{u_t}{\lambda} \partial_t - U_t$ , with analytic  $t$ -dependent families of Riemannian metrics  $\mathbf{g}_t$ , vector fields  $U_t$  and functions  $u_t$  on  $\mathcal{M}$  satisfying the initial*

conditions  $\mathbf{g}_0 = \mathbf{g}$ ,  $U_0 = U$ ,  $u_0 = u$ , and

$$\begin{aligned}\dot{\mathbf{g}}_0 &= -2\lambda_0 \mathbf{II}, \\ \dot{U}_0 &= u \operatorname{grad}^{\mathbf{g}}(\lambda_0) + \lambda_0 W(U), \\ \dot{u}_0 &= d\lambda_0(U).\end{aligned}$$

As an illustration, in Proposition 4.1 we provide an example in which  $W$  is a Codazzi tensor on  $(\mathcal{M}, \mathbf{g})$  and for which we can explicitly solve the corresponding system (3) for a constant function  $\lambda$ .

In Section 5 we will study to the problem of finding parallel spinors on the Lorentzian manifold in (1). For Riemannian manifolds, the corresponding Cauchy problem was studied by Ammann, Moroianu and Moroianu [1] in relation to the Cauchy problem for Ricci-flat manifolds. But, in contrast to the Riemannian situation, Lorentzian manifolds with parallel spinors are not necessarily Ricci-flat. Hence, for Lorentzian manifolds the Cauchy problem B(ii) for parallel null spinors in general is *not* a special case of the Cauchy problem for Lorentzian Ricci-flat metrics (which we review briefly in Section 2). However, since a parallel spinor  $\phi$  on a Lorentzian manifold induces a parallel *Dirac current*  $V_\phi$ , in the case when  $V_\phi$  is null, we can apply Theorem 1 instead. First we answer question A(ii): the parallel spinor  $\phi$  induces a spinor field  $\varphi$  on the Cauchy hypersurface  $\mathcal{M}$ , which satisfies the following *constraint conditions*

$$\begin{aligned}\nabla_X^S \varphi &= \frac{i}{2} W(X) \cdot \varphi, \quad \forall X \in T\mathcal{M}, \\ U_\varphi \cdot \varphi &= i u_\varphi \varphi,\end{aligned}\tag{4}$$

in which  $U_\varphi$  is the “*Riemannian*” *Dirac current* of  $\varphi$  defined by  $\mathbf{g}(U_\varphi, X) = -i(X \cdot \varphi, \varphi)$ ,  $u_\varphi = \sqrt{\mathbf{g}(U_\varphi, U_\varphi)} = \|\varphi\|^2$ , and  $W$  is the Weingarten operator of  $\mathcal{M} \subset (\overline{\mathcal{M}}, \overline{\mathbf{g}})$ . A spinor satisfying equations (4) with a symmetric endomorphism field  $W$  is called *generalised imaginary Killing spinor* or, more precisely, *imaginary W-Killing spinor*. We should mention that the case of *generalised real Killing spinors*, which correspond to parallel spinors for metrics of the form  $\overline{\mathbf{g}} = dr^2 + \mathbf{g}_r$ , was studied by Bär, Gauduchon and Moroianu [2].

In order to answer question B(ii) by applying Theorem 1, one checks that the data  $(W, U_\varphi)$  associated to an imaginary  $W$ -Killing spinor on  $(\mathcal{M}, \mathbf{g})$  satisfy the constraint conditions (2) for a parallel vector field. Then we can apply Theorem 1 and obtain:

**Theorem 2.** *Let  $(\mathcal{M}, \mathbf{g})$  be an analytic Riemannian spin manifold with an analytic  $\mathbf{g}$ -symmetric endomorphism field  $W$  and  $\varphi$  an imaginary  $W$ -Killing spinor on  $(\mathcal{M}, \mathbf{g})$ . Then  $(\mathcal{M}, \mathbf{g}, W, U_\varphi)$  satisfies the constraint conditions (2) and on the Lorentzian manifold  $(\overline{\mathcal{U}}, \overline{\mathbf{g}} = -\lambda^2 dt^2 + \mathbf{g}_t)$  obtained in Theorem 1, and with parallel null vector field  $V$ , there exists a parallel null spinor field  $\phi$  with Dirac current  $V$ . The parallel spinor  $\phi$  is obtained by parallel transport of  $\varphi$  along the lines  $t \mapsto (t, x)$ .*

Finally, in Section 6 we give examples of *complete* Riemannian metrics satisfying the constraint equations (2), including a metric on the 2-torus.

Our results in Theorems 1 and 2 are just the beginning of studying global hyperbolicity for manifolds with special holonomy and they suggest further questions:

- Can Theorems 1 and 2 be generalised to the smooth setting?
- Under which conditions do exist long term solutions to the Cauchy problems (B) that give a Lorentzian metrics on  $\overline{\mathcal{M}} = \mathbb{R} \times \mathcal{M}$ ?
- Provided there is a long term solution on  $\overline{\mathcal{M}} = \mathbb{R} \times \mathcal{M}$ , under which conditions is the resulting Lorentzian manifold globally hyperbolic?

- Is there a classification of Riemannian manifolds satisfying the constraint conditions (2) and (4)?

Having the analogous situation for the Lorentzian Einstein equation in mind, for which the work of Choquet-Bruhat [11] settled the problem in smooth case, it is very likely that the answer to the first question is positive. However, answers to these questions require techniques that are beyond the scope of this paper and have to be postponed to future research.

## 2. METRICS OF THE FORM $\bar{\mathbf{g}} = -\lambda^2 dt^2 + \mathbf{g}_t$

In the following, we consider product manifolds of the form  $\bar{\mathcal{M}} := \mathbb{R} \times \mathcal{M}$ , where  $\mathcal{M}$  is a smooth manifold of dimension  $n$ , with Lorentzian metrics

$$\bar{\mathbf{g}} = -\lambda^2 dt^2 + \mathbf{g}_t, \quad (5)$$

where  $\mathbf{g}_t$  is a  $t$ -dependent family of Riemannian metrics on  $\mathcal{M}$  and  $\lambda = \lambda(t, x)$  is a smooth function on  $\bar{\mathcal{M}}$ , the so-called *lapse function*. For a metric of the form (5) we fix a time-like unit vector field

$$T := \lambda^{-1} \partial_t.$$

In the following, by a bar we denote geometric objects defined by  $\bar{\mathbf{g}}$  such as the Levi-Civita connection  $\bar{\nabla}$ . Objects without bar come from  $\mathbf{g}_t$  and do depend on the parameter  $t \in \mathbb{R}$ . We will indicate this with an (upper or lower) index  $t$ . Vector fields  $U$  on  $\bar{\mathcal{M}}$  that are orthogonal to  $T$  (or, equivalently, tangent to  $\mathcal{M}$ ) can be considered in two ways: as sections of the bundle  $T^\perp \rightarrow \bar{\mathcal{M}}$ , i.e.,  $U \in \Gamma(T^\perp)$ , or as  $t$ -dependent sections of the bundle  $T\mathcal{M} \rightarrow \mathcal{M}$ , i.e.,  $U_t \in \Gamma(T\mathcal{M})$  for all  $t$ . Similarly, we will treat sections of tensor bundles of  $T^\perp \rightarrow \bar{\mathcal{M}}$ . Finally, when useful, we write functions on  $\bar{\mathcal{M}}$  as  $t$ -dependent families of functions on  $\mathcal{M}$ , i.e.,  $u_t = u(t, \cdot)$ . Vector fields (or functions) on  $\mathcal{M}$  and their lifts to  $\bar{\mathcal{M}}$  are denoted by the same symbol.

The gradient of the lapse function is related to the derivative of  $T$  as follows,

$$\bar{\nabla}_T T = \text{grad}^t(\log \lambda), \quad \bar{\nabla}_{\partial_t} T = \text{grad}^t(\lambda),$$

where  $\text{grad}^t$  denotes the gradient with respect to the metric  $\mathbf{g}_t$ . For  $X, Y \in T\mathcal{M}$  denote by

$$\Pi_t(X, Y) := -\bar{\mathbf{g}}(\bar{\nabla}_X T, Y)$$

the second fundamental form of  $(\mathcal{M}, \mathbf{g}_t) \subset (\bar{\mathcal{M}}, \bar{\mathbf{g}})$ , i.e., we have

$$\bar{\nabla}_X Y = \nabla_X^t Y - \Pi_t(X, Y)T$$

in which  $\nabla^t$  denotes the Levi-Civita connection of  $\mathbf{g}_t$ . The dual of the second fundamental form is the *Weingarten operator* defined by

$$\Pi_t(X, Y) = \mathbf{g}_t(W_t(X), Y),$$

i.e.,  $W_t = -\bar{\nabla}T|_{T\mathcal{M}}$ . The second fundamental form is computed in terms of  $\mathbf{g}_t$  as

$$\Pi_t(X, Y) = -\frac{1}{2}\lambda^{-1}(\mathcal{L}_{\partial_t}\mathbf{g}_t)(X, Y),$$

where  $\mathcal{L}$  denotes the Lie derivative. Hence, extending  $X$  and  $Y$  independent of  $t$  we have

$$\Pi_t(X, Y) = -\frac{1}{2}\lambda^{-1}\partial_t\mathbf{g}_t(X, Y) =: -\frac{1}{2}\lambda^{-1}\dot{\mathbf{g}}_t(X, Y) \quad (6)$$

$$\dot{\Pi}_t(X, Y) = \frac{1}{2}\lambda^{-1}((\log \lambda)\dot{\mathbf{g}}_t(X, Y) - \ddot{\mathbf{g}}_t(X, Y)), \quad (7)$$

where the dot denotes the partial  $t$  derivative. Moreover, for a symmetric  $(2,0)$ -tensor field  $h$  and a one form  $\mu$  we use the notation

$$\begin{aligned} d^\nabla h(X, Y, Z) &:= (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \\ \mu \wedge h(X, Y, Z) &:= \mu(X)h(Y, Z) - \mu(Y)h(X, Z). \end{aligned}$$

The skew symmetric derivative  $d^\nabla h$  satisfies the first Bianchi identity

$$d^\nabla h(X, Y, Z) + d^\nabla h(Y, Z, X) + d^\nabla h(Z, X, Y) = 0. \quad (8)$$

The trace of  $d^\nabla h(X, \cdot, \cdot)$  is given by the divergence and the trace of  $h$ , via

$$\text{tr}_{\mathbf{g}_t} d^\nabla h(X, \cdot, \cdot) = \text{div}^t h(X) + d \text{tr}_{\mathbf{g}_t} h(X). \quad (9)$$

The curvature  $\overline{\mathbf{R}}$  of  $\overline{\mathbf{g}}$  defined as  $\overline{\mathbf{R}}(U, V) := [\overline{\nabla}_X, \overline{\nabla}_Y] - \overline{\nabla}_{[U, V]}$  is linked to the curvature  $\mathbf{R}^t$  of  $\mathbf{g}_t$  by the *Gauß equation*

$$\overline{\mathbf{R}}(X, Y, Z, U) = \mathbf{R}_t(X, Y, Z, U) - \Pi_t(X, Z)\Pi_t(Y, U) + \Pi_t(X, U)\Pi_t(Y, Z), \quad (10)$$

with  $X, Y, Z, U \in T\mathcal{M}$ , the *Codazzi equation*

$$\begin{aligned} \overline{\mathbf{R}}(X, Y, Z, T) &= d^\nabla \Pi_t(X, Y, Z) \\ &= -\frac{1}{2\lambda} \left( d^\nabla \dot{\mathbf{g}}_t(X, Y, Z) - (d \log \lambda) \wedge \dot{\mathbf{g}}_t(X, Y, Z) \right), \end{aligned} \quad (11)$$

and the following formula, sometimes called *Mainardi equation*,

$$\overline{\mathbf{R}}(X, T, T, Y) = \Pi_t(X, W_t(Y)) + \frac{1}{\lambda} \left( \dot{\Pi}_t(X, Y) + \text{Hess}^t(\lambda)(X, Y) \right), \quad (12)$$

where  $\text{Hess}^t(f) = \nabla^t df$  denotes the Hessian of a function with respect to the metric  $\mathbf{g}_t$ , and  $W_t$  is the Weingarten operator. Indeed, since  $\overline{\nabla}_T T = \text{grad}^t(\log \lambda)$ , we have

$$\begin{aligned} \overline{\mathbf{R}}(X, T, T, Y) &= \text{Hess}^t(\log \lambda)(X, Y) - (T(\overline{\mathbf{g}}(\overline{\nabla}_X T, Y)) - \overline{\mathbf{g}}(\overline{\nabla}_{[T, X]} T, Y)) + \overline{\mathbf{g}}(\overline{\nabla}_X T, \overline{\nabla}_T Y) \\ &= \text{Hess}^t(\log \lambda)(X, Y) - \mathcal{L}_T(\overline{\mathbf{g}}(\overline{\nabla} T, \cdot))(X, Y) + \overline{\mathbf{g}}(\overline{\nabla}_X T, \overline{\nabla}_T Y), \end{aligned}$$

which proves (12), when taking into account that

$$\begin{aligned} \mathcal{L}_T(\overline{\mathbf{g}}(\overline{\nabla} T, \cdot))(X, Y) &= -\lambda^{-1} \mathcal{L}_{\partial_t} \Pi_t(X, Y) + X(\lambda^{-1})Y(\lambda) \\ &= -\frac{\dot{\Pi}_t(X, Y)}{\lambda} - X(\log \lambda)Y(\log \lambda). \end{aligned}$$

and that

$$\text{Hess}^t(\log \lambda) + d(\log \lambda)^2 = \frac{1}{\lambda} \text{Hess}^t(\lambda).$$

For the Ricci curvature

$$\overline{\text{Ric}} = -\overline{\mathbf{R}}(T, \cdot, \cdot, T) + \sum_{i=1}^n \overline{\mathbf{R}}(E_i, \cdot, \cdot, E_i)$$

equation (9) gives

$$\overline{\text{Ric}}(T, T) = \|\Pi_t\|_{\mathbf{g}_t}^2 + \frac{1}{\lambda} \left( \text{tr}^t(\dot{\Pi}_t) + \Delta_t(\lambda) \right) \quad (13)$$

$$\overline{\text{Ric}}(X, T) = d(\text{tr}^t \Pi_t)(X) + \text{div}^t \Pi_t(X) \quad (14)$$

and

$$\begin{aligned} \overline{\text{Ric}}(X, Y) &= \text{Ric}^t(X, Y) + \text{tr}^t(\Pi_t)\Pi_t(X, Y) - 2\Pi_t(X, W_t(Y)) \\ &\quad - \frac{1}{\lambda} \left( \dot{\Pi}_t(X, Y) + \text{Hess}^t(\lambda)(X, Y) \right), \end{aligned} \quad (15)$$

in which  $\text{tr}^t$  is the trace,  $\text{div}^t$  the divergence and  $\Delta_t = \text{tr}^t(\text{Hess}^t)$  the Laplacian, all with respect to  $\mathbf{g}_t$ .  $\|\text{II}\|_{\mathbf{g}_t}^2$  is the norm with respect to  $\mathbf{g}_t$ , which is equal to  $\text{tr}^t(W_t^2)$ . Finally, for the scalar curvature we get

$$\overline{\text{scal}} = \text{scal}^t + (\text{tr}^t(\text{II}_t))^2 - 3\|\text{II}_t\|_{\mathbf{g}_t}^2 - \frac{2}{\lambda} \left( \text{tr}^t(\dot{\text{II}}_t) + \Delta_t(\lambda) \right). \quad (16)$$

These formulae give us the well known constraint and evolution equations for Ricci flat Lorentzian metrics (see [3] for a review). In fact, the Lorentzian metric (5) is Ricci flat if and only if, the Riemannian metrics  $\mathbf{g}_t$  together with the symmetric bilinear form  $\text{II}_t$  satisfy the *constraint equations*

$$\begin{aligned} \text{scal}^t &= \|\text{II}_t\|_{\mathbf{g}_t}^2 - \text{tr}^t(\text{II}_t)^2 \\ d \text{tr}^t \text{II}_t &= -\text{div}^t \text{II}_t, \end{aligned} \quad (17)$$

which follow from setting (13), (16) and (14) to zero, and the *evolutions equation* for  $\text{II}_t$ ,

$$\dot{\text{II}}_t(X, Y) = \lambda \left( \text{Ric}^t(X, Y) + \text{tr}^t(\text{II}_t) \text{II}_t(X, Y) - 2\text{II}_t(X, W_t(Y)) \right) - \text{Hess}^t(\lambda)(X, Y), \quad (18)$$

which comes from equation (15) and in which the dot denotes the  $t$ -derivative. Rewriting this equation in terms of  $\mathbf{g}_t$  using (6), it becomes an evolution equation for  $\mathbf{g}_t$ , namely

$$\begin{aligned} \dot{\mathbf{g}}_t(X, Y) &= \left( (\log \lambda) - \frac{\text{tr}^t(\dot{\mathbf{g}}_t)}{2} \right) \dot{\mathbf{g}}_t(X, Y) - 2\lambda \dot{\mathbf{g}}_t(X, W_t(Y)) \\ &\quad + 2\lambda \text{Hess}^t(\lambda)(X, Y) - 2\lambda^2 \text{Ric}^t(X, Y). \end{aligned} \quad (19)$$

One can check that the constraint equations (17) are preserved under this flow (see for example [13, p. 438]). For analytic data (initial conditions and  $\lambda$ ) one can apply the Cauchy-Kowalevski Theorem (an excellent reference is [10]) in order to obtain a unique analytic solution. That this can be done also for smooth data is the result of fundamental work by Y. Choquet-Bruhat in [11].

### 3. CONSTRAINT AND EVOLUTION EQUATIONS FOR PARALLEL NULL VECTORS

In this section we study the problem of extending a given Riemannian manifold to a Lorentzian manifold of the form (5) under the condition that the metric  $\overline{\mathbf{g}}$  admits a *parallel null vector field*.

In the following let  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  be a time-oriented Lorentzian manifold of the form (5) with time orientation given by  $T$  and let  $V$  be a vector field. We denote the time component of  $V$  by

$$u := -\overline{\mathbf{g}}(T, V). \quad (20)$$

If  $V$  is null, i.e.,  $\overline{\mathbf{g}}(V, V) = 0$ , then  $u > 0$ . The vector fields  $T$  and  $V$  define a global space-like vector field  $U$  on  $\overline{\mathcal{M}}$  tangent to  $\mathcal{M}$  by projecting  $-V$  along  $T$  onto  $\mathcal{M}$ ,

$$U := uT - V, \quad \text{i.e.,} \quad V = uT - U, \quad (21)$$

yielding  $\overline{\mathbf{g}}(U, U) = u^2$ . On the other hand, if  $U$  is a nowhere vanishing vector field on  $\overline{\mathcal{M}}$  which is tangent to  $\mathcal{M}$  in any point, then  $U$  is space-like,  $u := \sqrt{\overline{\mathbf{g}}(U, U)} > 0$  and  $V := uT - U$  is null and future directed with  $u = -\overline{\mathbf{g}}(V, T)$ . Again we consider  $U$  and  $u$  as  $t$ -dependent families of vector fields  $U_t = U(t, \cdot)$  and functions  $u_t = u(t, \cdot)$  on  $\mathcal{M}$ . Now we observe the following:

**Lemma 3.1.** *Let  $V$  be a null vector field on  $\overline{\mathcal{M}}$  and  $\overline{X} \in T\overline{\mathcal{M}}$ . Then  $\overline{\nabla}_{\overline{X}} V = 0$  if and only if  $\text{pr}_{T^\perp}(\overline{\nabla}_{\overline{X}} V) = 0$ .*

*Proof.* Let  $V = uT - U$  as in (21). Since  $V$  is null we have

$$0 = \bar{\mathbf{g}}(\bar{\nabla}_{\bar{X}}V, V) = u\bar{\mathbf{g}}(\bar{\nabla}_{\bar{X}}V, T) - \bar{\mathbf{g}}(\bar{\nabla}_{\bar{X}}V, U).$$

Hence, under the assumptions that  $\text{pr}_{T^\perp}(\bar{\nabla}_{\bar{X}}V) = 0$  we have, in particular, that  $\bar{\mathbf{g}}(\bar{\nabla}_{\bar{X}}V, U) = 0$  which proves the claim.  $\square$

In the following, we will denote by  $\dot{u}_t$  and  $\dot{U}_t$  the Lie derivatives

$$\dot{u}_t := \partial_t(u) \text{ and } \dot{U}_t := \mathcal{L}_{\partial_t}U_t = [\partial_t, U_t].$$

**Proposition 3.1.** *Let  $V$  a future directed null vector field,  $U_t$  the corresponding space-like projection onto  $\mathcal{M}$  as defined in (21) and  $u_t = \sqrt{\bar{\mathbf{g}}(U_t, U_t)} = -\bar{\mathbf{g}}(T, V)$ .*

(1) *Let  $X \in T\mathcal{M}$ . Then  $\bar{\nabla}_X V = 0$  is equivalent to*

$$\nabla_X^t U_t = -u_t W_t(X). \quad (22)$$

*In particular, the condition  $\bar{\nabla}_X V = 0$  implies  $du_t(X) = -\Pi_t(U_t, X)$ .*  
 (2)  *$\bar{\nabla}_{\partial_t} V = 0$  if and only if*

$$\bar{\nabla}_{\partial_t} U_t = \dot{u}_t T + u_t \text{grad}^t \lambda, \quad (23)$$

*which is equivalent to*

$$\dot{U}_t = u_t \text{grad}^t(\lambda) + \lambda W_t(U_t). \quad (24)$$

*The condition  $\bar{\nabla}_{\partial_t} V = 0$  implies  $\dot{u}_t = d\lambda(U)$ .*

*Proof.* Using  $V = u_t T - U$  and  $\bar{\nabla}_{\partial_t} T = \text{grad}^t(\lambda)$ , we obtain

$$\begin{aligned} 0 = \bar{\nabla}_{\bar{X}} V &= du(\bar{X})T + u\bar{\nabla}_{\bar{X}}T - \bar{\nabla}_{\bar{X}}U \\ &= \begin{cases} (du_t(X) + \Pi_t(X, U_t))T - u_t W_t(X) - \nabla_X^t U_t, & \text{if } \bar{X} = X \in T\mathcal{M}, \\ \dot{u}_t T + u_t \text{grad}^t(\lambda) - \bar{\nabla}_{\partial_t} U_t, & \text{if } \bar{X} = \partial_t. \end{cases} \end{aligned}$$

Lemma 3.1 shows that the first equation is equivalent to (22). The second one is equivalent to (23). Again, applying Lemma 3.1 and

$$\bar{\nabla}_{\partial_t} U = [\partial_t, U] + \bar{\nabla}_U \partial_t = \dot{U}_t + \bar{\nabla}_U \lambda T = \dot{U}_t + d\lambda(U)T - \lambda W_t(U_t) \quad (25)$$

we get the equivalence of (23) and (24). With Lemma 3.1 equation (22) implies  $du_t(X) = -\Pi_t(U_t, X)$  and (23) and (25) imply  $\dot{u}_t = d\lambda(U)$ .  $\square$

Now we drop the assumption that  $V$  is a *null* vector field for a moment.

**Lemma 3.2.** *Let  $V$  be a vector field on  $(\bar{\mathcal{M}}, \bar{\mathbf{g}})$ . Then  $V$  is parallel for  $\bar{\mathbf{g}}$  if and only if the following conditions are satisfied:*

$$\bar{\mathbf{R}}(\partial_t, X)V = 0, \text{ for all } X \in T\mathcal{M}, \quad (26)$$

$$\bar{\nabla}_{\partial_t} \bar{\nabla}_{\partial_t} V = 0, \quad (27)$$

$$\bar{\nabla}_X V|_{\{0\} \times \mathcal{M}} = 0, \text{ for all } X \in T\mathcal{M}|_{\{0\} \times \mathcal{M}} \quad (28)$$

$$\bar{\nabla}_{\partial_t} V|_{\{0\} \times \mathcal{M}} = 0. \quad (29)$$



*Proof.* Clearly, if  $V$  is parallel, all the conditions follow immediately. Thus, let us assume the four conditions. Firstly, the equation (27) shows that the vector field  $\bar{\nabla}_{\partial_t} V$  is parallel transported along the curves  $t \mapsto (t, x)$ . Hence, because of the initial condition (29), we get that  $\bar{\nabla}_{\partial_t} V = 0$  everywhere. Using this, equation (26) gives that

$$0 = \bar{R}(\partial_t, X)V = \bar{\nabla}_{\partial_t} \bar{\nabla}_X V - \bar{\nabla}_X \bar{\nabla}_{\partial_t} V - \bar{\nabla}_{[\partial_t, X]} V = \bar{\nabla}_{\partial_t} \bar{\nabla}_X V \quad (30)$$

whenever  $X$  is the lift of a vector field of  $\mathcal{M}$  to  $\bar{\mathcal{M}}$ , i.e., such that  $[\partial_t, X] = 0$ . This shows that  $\bar{\nabla}_X V$  is parallel transported along all  $t \mapsto (t, x)$ . Since we have assumed that  $\bar{\nabla}_X V = 0$  along the initial manifold  $\mathcal{M}$ , it also shows that  $V$  is parallel on  $\bar{\mathcal{M}}$ .  $\square$

Now we will study the equations (26) and (27) further.

**Lemma 3.3.** *Let  $(\bar{\mathcal{M}}, \bar{g})$  be a Lorentzian manifold as in (5).*

(1) *There exists a vector field  $V \in \Gamma(T\bar{\mathcal{M}})$  such that*

$$\bar{R}(X, Y)V = 0 \quad \text{for all } X, Y \in T\mathcal{M},$$

*if and only if there is a smooth family of vector fields  $\{U_t\}_{t \in \mathbb{R}}$  and functions  $\{u_t\}_{t \in \mathbb{R}}$  on  $\mathcal{M}$ , such that*

$$R^t(X, Y, Z, U_t) = u_t d^{\nabla^t} \Pi_t(X, Y, Z) + \Pi_t(X, Z) \Pi_t(Y, U_t) - \Pi_t(Y, Z) \Pi_t(X, U_t), \quad (31)$$

*for all  $X, Y, Z \in T\mathcal{M}$ .*

(2) *There exists a vector field  $V \in \Gamma(T\bar{\mathcal{M}})$  with*

$$\bar{R}(T, X)V = 0 \quad \text{for all } X \in T\mathcal{M},$$

*if and only if there is a smooth family of vector fields  $\{U_t\}_{t \in \mathbb{R}}$  and functions  $\{u_t\}_{t \in \mathbb{R}}$  on  $\mathcal{M}$ , such that*

$$u_t \dot{\Pi}_t(X, Y) = \lambda(d^{\nabla^t} \Pi_t)(U_t, Y, X) - u_t \lambda \Pi_t(X, W_t(Y)) - u_t \text{Hess}^t(\lambda)(X, Y), \quad (32)$$

*or equivalently,*

$$\begin{aligned} u_t \ddot{g}_t(X, Y) &= \lambda^2 d^{\nabla^t} \left( \frac{\dot{g}_t}{\lambda} \right) (U_t, Y, X) - u_t \lambda \dot{g}_t(X, W_t(Y)) \\ &\quad + u_t (\log \lambda) \dot{g}_t(X, Y) + 2u_t \lambda \text{Hess}^t(\lambda)(X, Y), \end{aligned} \quad (33)$$

*for all  $X, Y \in T\mathcal{M}$ .*

*Proof.* We again use the relation between  $V$  and  $(U_t, u_t)$  given by  $V = u_t T - U_t$  with  $u_t = -\bar{g}(V, T)$ , where  $T = \lambda^{-1} \partial_t$  is the time like unit vector field. Then, for vectors  $\bar{X} \in T\bar{\mathcal{M}}$  and  $Y \in T\mathcal{M}$  we have

$$\bar{R}(\bar{X}, Y)V = u_t \bar{R}(\bar{X}, Y)T - \bar{R}(\bar{X}, Y)U_t.$$

Hence,  $\bar{R}(\bar{X}, Y)V = 0$  is equivalent to the equations

$$\begin{aligned} \bar{R}(\bar{X}, Y, U_t, T) &= 0 \\ -u_t \bar{R}(\bar{X}, Y, Z, T) + \bar{R}(\bar{X}, Y, Z, U_t) &= 0 \end{aligned} \quad (34)$$

for all  $Y, Z \in T\mathcal{M}$  and  $\bar{X} \in T\bar{\mathcal{M}}$ . Note that in case  $u_t$  has no zeros, the second equation implies the first as it holds for all  $Z \in T\mathcal{M}$  including  $U_t$ . Setting  $\bar{X} = X \in T\mathcal{M}$ , the Gauß and Codazzi equations (10) and (11) show that (34) is equivalent to (31).



On the other hand for  $\overline{X} = T$ , the Codazzi and Mainardi equations (11) and (12) show that (34) is equivalent to (32). Computing  $\dot{\Pi}_t$  as

$$\dot{\Pi}_t(X, Y) = \frac{1}{2\lambda} \left( \log \lambda \, \dot{\mathbf{g}}_t(X, Y) - \ddot{g}_t(X, Y) \right),$$

using (6), shows that (32), when written out in terms of  $\dot{\mathbf{g}}_t$ , just becomes (33).  $\square$

Next, we look at the second derivative of a null vector field in the  $t$ -direction.

**Lemma 3.4.** *There exists a vector field  $V$  on  $\overline{\mathcal{M}}$  with*

$$\overline{\nabla}_{\partial_t} \overline{\nabla}_{\partial_t} V = 0,$$

*if and only if there exist a smooth family of vector fields  $U_t$  and functions  $u_t$  on  $\mathcal{M}$ , such that*

$$\begin{aligned} \ddot{U}_t &= \lambda \left( [\partial_t, W_t(U_t)] + W_t(\dot{U}_t) - \lambda W_t^2(U_t) \right) \\ &\quad + u_t \left( [\partial_t, \text{grad}^t \lambda] - \lambda W_t(\text{grad}^t \lambda) \right) + \dot{\lambda}_t W_t(U_t) + (2\dot{u}_t - d\lambda(U_t)) \text{grad}^t \lambda \end{aligned} \quad (35)$$

*and*

$$\ddot{u}_t = \mathbf{g}_t([\partial_t, \text{grad}^t \lambda], U_t) + 2d\lambda(\dot{U}_t) - 3\lambda d\lambda(W_t(U_t)) - u_t \|\text{grad}^t \lambda\|_t^2. \quad (36)$$

*Proof.* Note that  $\overline{\nabla}_{\partial_t} T = \text{grad}^t \lambda$  and

$$\overline{\nabla}_{\partial_t} X = d\lambda(X)T + \dot{X}_t - \lambda W_t(X),$$

for  $X \in \Gamma(T\mathcal{M})$  but possibly depending on  $t$ . Using this, for  $V = u_t T - U_t$  we compute

$$\begin{aligned} \overline{\nabla}_{\partial_t} V &= \dot{u}_t T + u_t \text{grad}^t(\lambda) - \overline{\nabla}_{\partial_t} U_t \\ &= (\dot{u}_t - d\lambda(U_t))T + u_t \text{grad}^t(\lambda) - \dot{U}_t + \lambda W_t(U_t). \end{aligned}$$

Applying  $\overline{\nabla}_{\partial_t}$  to this, we get

$$\begin{aligned} \overline{\nabla}_{\partial_t} \overline{\nabla}_{\partial_t} V &= \left( \ddot{u}_t - \partial_t(d\lambda(U_t)) + u_t \|\text{grad}^t \lambda\|_t^2 - d\lambda(\dot{U}_t) + \lambda d\lambda(W_t(U_t)) \right) T \\ &\quad - \ddot{U}_t + \lambda \left( [\partial_t, W_t(U_t)] + W_t(\dot{U}_t) - \lambda W_t^2(U_t) \right) \\ &\quad + u_t \left( [\partial_t, \text{grad}^t \lambda] - \lambda W_t(\text{grad}^t \lambda) \right) + \dot{\lambda}_t W_t(U_t) + (2\dot{u}_t - d\lambda(U_t)) \text{grad}^t \lambda. \end{aligned}$$

Setting  $\overline{\nabla}_{\partial_t} \overline{\nabla}_{\partial_t} V = 0$  already implies (35). Computing

$$\begin{aligned} \partial_t(d\lambda(U_t)) &= \overline{\mathbf{g}}(\overline{\nabla}_{\partial_t} \overline{\text{grad}} \lambda, U) + \overline{\mathbf{g}}(\overline{\text{grad}} \lambda, \overline{\nabla}_{\partial_t} U) \\ &= \mathbf{g}_t([\partial_t, \text{grad}^t \lambda], U_t) + d\lambda([\partial_t, U_t]) - 2\lambda d\lambda(W_t(U_t)), \end{aligned}$$

we obtain also (36).  $\square$

For later purposes, we will now record the dualisation of formula (35).

**Lemma 3.5.** *Equation (35) is equivalent to*

$$\begin{aligned} \mathbf{g}_t(\ddot{U}_t, X) &= -\frac{1}{2} \ddot{\mathbf{g}}_t(U_t, X) - \dot{\mathbf{g}}_t(\dot{U}_t, X) - \frac{\lambda}{2} \dot{\mathbf{g}}_t(W_t(U_t), X) \\ &\quad + u_t \mathbf{g}_t([\partial_t, \text{grad}^t \lambda], X) + \frac{u_t}{2} \dot{\mathbf{g}}_t(\text{grad}^t \lambda, X) + (2\dot{u}_t - d\lambda(U_t)) d\lambda(X), \end{aligned} \quad (37)$$

for any  $X \in T\mathcal{M}$ .

*Proof.* We compute the first term in the right hand side of (35) as

$$\begin{aligned}
\lambda \mathbf{g}_t([\partial_t, W_t(U_t)], X) &= \lambda \bar{\mathbf{g}}(\bar{\nabla}_{\partial_t} W_t(U_t) - \bar{\nabla}_{W_t(U_t)} \partial_t, X) \\
&= \lambda \partial_t(\Pi_t(U_t, X)) - \lambda \bar{\mathbf{g}}(W_t(U_t), \bar{\nabla}_{\partial_t} X) + \lambda^2 \Pi_t(W_t(U_t), X) \\
&= -\frac{\lambda}{2} \partial_t \left( \frac{1}{\lambda} \dot{\mathbf{g}}_t(U_t, X) \right) + 2\lambda^2 \Pi_t(W_t(U_t), X) \\
&= \frac{1}{2} \partial_t(\log \lambda) \dot{\mathbf{g}}_t(U_t, X) - \frac{1}{2} \left( \ddot{\mathbf{g}}_t(U_t, X) + \dot{\mathbf{g}}_t(\dot{U}_t, X) \right) - \lambda \dot{\mathbf{g}}_t(W_t(U_t), X).
\end{aligned}$$

Hence, all of (35) gives

$$\begin{aligned}
\mathbf{g}_t(\ddot{U}_t, X) &= \frac{1}{2} \partial_t(\log \lambda) \dot{\mathbf{g}}_t(U_t, X) - \frac{1}{2} \left( \ddot{\mathbf{g}}_t(U_t, X) + \dot{\mathbf{g}}_t(\dot{U}_t, X) \right) - \lambda \dot{\mathbf{g}}_t(W_t(U_t), X) \\
&\quad - \frac{1}{2} \dot{\mathbf{g}}_t(\dot{U}_t, X) + \frac{\lambda}{2} \dot{\mathbf{g}}_t(W_t(U_t), X) - \frac{1}{2} \partial_t(\log \lambda) \dot{\mathbf{g}}_t(U_t, X) \\
&\quad + u_t \left( \mathbf{g}_t([\partial_t, \text{grad}^t \lambda], X) + \frac{1}{2} \dot{\mathbf{g}}_t(\text{grad}^t \lambda, X) \right) + (2\dot{u}_t - d\lambda(U_t)) d\lambda(X),
\end{aligned}$$

which proves the Lemma.  $\square$

Using the previous Lemmas, we obtain

**Theorem 3.1.** *Let  $(\mathcal{M}, \mathbf{g}, W, U)$  be a Riemannian manifold together with a field of  $\mathbf{g}$ -symmetric endomorphisms  $W$ , with corresponding symmetric bilinear form  $\Pi := \mathbf{g}(W\cdot, \cdot)$ , and a vector field  $U$  satisfying the following constraint equations*

$$\nabla^{\mathbf{g}} U + uW = 0, \quad (38)$$

where  $u^2 := \mathbf{g}(U, U)$ . Then, for any positive smooth function  $\lambda$  on  $\mathbb{R} \times \mathcal{M}$ , a triple  $(\mathbf{g}_t, U_t, u_t)$  of smooth one-parameter families of Riemannian metrics, vector fields and functions on  $\mathcal{M}$  defines a Lorentzian metric

$$\bar{\mathbf{g}} = -\lambda^2 dt^2 + \mathbf{g}_t$$

on an open neighbourhood  $\bar{U}(\{0\} \times \mathcal{M}) \subset \mathbb{R} \times \mathcal{M}$  with parallel null vector field

$$V = \frac{u_t}{\lambda} \partial_t - U_t,$$

if and only if  $\mathbf{g}_t$ ,  $U_t$  and  $u_t$  satisfy the following system of PDEs on  $\bar{U}$ ,

$$\begin{aligned}
\ddot{\mathbf{g}}_t(X, Y) &= \frac{\lambda^2}{u_t} d^{\nabla^t} \left( \frac{\dot{\mathbf{g}}_t}{\lambda} \right) (U_t, Y, X) + \frac{1}{2} \dot{\mathbf{g}}_t(X, \mathbf{g}_t^\sharp(Y)) + (\log \lambda) \dot{\mathbf{g}}_t(X, Y) \\
&\quad + 2\lambda \text{Hess}^t(\lambda)(X, Y),
\end{aligned} \quad (39)$$

$$\begin{aligned}
\mathbf{g}_t(\ddot{U}_t, X) &= -\frac{\lambda^2}{2u_t} d^{\nabla^t} \left( \frac{\dot{\mathbf{g}}_t}{\lambda} \right) (U_t, X, U_t) - \dot{\mathbf{g}}_t(\dot{U}_t, X) - \frac{\log \lambda}{2} \dot{\mathbf{g}}_t(U_t, X) \\
&\quad - \lambda \text{Hess}^t(\lambda)(U_t, X) + u_t \mathbf{g}_t([\partial_t, \text{grad}^t \lambda], X) + \frac{u_t}{2} \dot{\mathbf{g}}_t(\text{grad}^t \lambda, X) \\
&\quad + (2\dot{u}_t - d\lambda(U_t)) d\lambda(X),
\end{aligned} \quad (40)$$

$$\ddot{u}_t = \mathbf{g}_t([\partial_t, \text{grad}^t \lambda], U_t) + 2d\lambda(\dot{U}_t) + \frac{3}{2} \dot{\mathbf{g}}_t(\text{grad}^t(\lambda), U_t) - u_t \|\text{grad}^t \lambda\|_{\mathbf{g}_t}^2, \quad (41)$$

with the initial conditions

$$\begin{aligned}
\mathbf{g}_0 &= \mathbf{g}, \\
\dot{\mathbf{g}}_0 &= -2\lambda_0 \mathbf{II}, \\
U_0 &= U, \\
\dot{U}_0 &= u \operatorname{grad}^{\mathbf{g}}(\lambda_0) + \lambda_0 W(U), \\
u_0 &= u, \\
\dot{u}_0 &= d\lambda_0(U).
\end{aligned} \tag{42}$$

Here,  $\dot{\mathbf{g}}_t^\sharp$  denotes the metric dual of  $\dot{\mathbf{g}}_t$ , i.e.,  $\mathbf{g}_t(X, \dot{\mathbf{g}}_t^\sharp(Y)) = \dot{\mathbf{g}}_t(X, Y)$  and in terms of the Weingarten operator  $W_t$  of  $(\mathcal{M}, \mathbf{g}_t)$  we have  $\dot{\mathbf{g}}_t(X, \dot{\mathbf{g}}_t^\sharp(Y)) = 4\lambda^2 \mathbf{g}_t(W_t(X), W_t(Y)) = -2\lambda \dot{\mathbf{g}}_t(W_t(X), Y)$ .

*Proof.* Let  $(\mathbf{g}_t, U_t, u_t)$  be a triple of one-parameter families of Riemannian metrics, vector fields and functions on  $\mathcal{M}$ .

First, assume that the Lorentzian metric  $\bar{\mathbf{g}}$  defined by  $\lambda$  and  $\mathbf{g}_t$  admits a parallel null vector field  $V$  defined by  $\lambda$ ,  $U_t$  and  $u_t$ . Since  $V$  is null, we have  $\mathbf{g}_t(U_t, U_t) = u_t^2 > 0$ . Moreover, it implies  $\bar{\mathbf{R}}(T, X)V = 0$  for all  $X \in T\mathcal{M}$  and hence, by (33) in Lemma 3.3 and  $u_t > 0$  we obtain equation (39). Moreover, as  $V$  is parallel, Lemma 3.4 gives us (36) which is nothing else than (41). Combining Lemma 3.4 and Lemma 3.5 shows that also (37) holds. Now using the obtained (39) in order to substitute the term  $\ddot{\mathbf{g}}_t(U_t, X)$  in (37), we obtain (40).

Now assume that, for a given  $\lambda$ , the triple  $(\mathbf{g}_t, U_t, u_t)$  of Riemannian metrics, vector fields and functions on  $\mathcal{M}$  solves equations (39, 40, 41) with the given initial conditions. Using this solution, we define the vector field  $V = u_t T - U_t$  on  $\bar{\mathcal{U}}$ , with  $T = \lambda^{-1} \partial_t$ . Then, by equation (33) in Lemma 3.3, equation (39) shows that  $V$  satisfies

$$\bar{\mathbf{R}}(T, X)V = 0$$

for all  $X \in T\mathcal{M}$ . We can use equation (39) to substitute  $-\frac{1}{2}\ddot{\mathbf{g}}_t(U_t, X)$  into equation (40). This equation becomes exactly equation (37) in Lemma 3.5. On the other hand, equation (41) is just (36), and hence, by Lemma 3.4 we obtain that

$$\bar{\nabla}_{\partial_t} \bar{\nabla}_{\partial_t} V = 0.$$

The initial condition  $u_0 = u$  together with the constraint  $\mathbf{g}(U, U) - u^2 = 0$  show that the vector field  $V|_{\mathcal{M}} = u_0 T - U_0 \in T\bar{\mathcal{U}}|_{\mathcal{M}}$  has constant length 0 along the initial manifold  $\mathcal{M} \simeq \{0\} \times \mathcal{M}$ . Then, as in Lemma 3.1, we have

$$0 = X\bar{\mathbf{g}}(V, V)|_{t=0} = 2\bar{\mathbf{g}}(\bar{\nabla}_X V, V)|_{t=0} = 2u_0\bar{\mathbf{g}}(\bar{\nabla}_X V|_{t=0}, T_0)|_{t=0} - 2\bar{\mathbf{g}}(\bar{\nabla}_X V|_{t=0}, U_0)$$

which shows that  $\bar{\nabla}_X V|_{t=0} = 0$  if  $\operatorname{pr}_{T^\perp}(\bar{\nabla}_X V|_{t=0}) = 0$ . But by Proposition 3.1 we have that  $\operatorname{pr}_{T^\perp}(\bar{\nabla}_X V|_{t=0}) = 0$ , and hence  $\bar{\nabla}_X V|_{t=0} = 0$ , if

$$\nabla_X^0 U_0 = -u_0 W_0(X),$$

which is the constraint condition (38). Thus, along the initial manifold  $\mathcal{M}$  we have

$$\bar{\nabla}_X V|_{t=0} = 0.$$

Finally, we show that, along the initial manifold  $\mathcal{M}$  we also have  $\bar{\nabla}_{\partial_t} V|_{t=0} = 0$ . As computed above, we have

$$\bar{\nabla}_{\partial_t} V|_{t=0} = (\dot{u}_0 - d\lambda(U_0))T_0 + u_0 \operatorname{grad}^0(\lambda) - \dot{U}_0 + \lambda W_0(U_0) = 0,$$

along  $\mathcal{M}$ , because of the initial conditions (42). Hence, all assumptions of Lemma 3.2 are satisfied and we obtain that  $V$  is parallel on  $\bar{\mathcal{U}}$ . But since  $V$  has constant length 0 along the initial surface  $\mathcal{M}$  and is parallel, it has constant length 0 everywhere.  $\square$

**Remark 3.1.** We observe that the Cauchy-Kowalevski theorem can be applied to the PDE system in Theorem 3.1, provided that all initial data are assumed as analytic. It guarantees existence and uniqueness of solutions to the evolution equations (39) - (41) for the given constraints and initial conditions for data on  $\mathcal{M}$  in an open neighbourhood of  $\{0\} \times \mathcal{M}$ . However, we do not know whether the solution  $\mathbf{g}_t$  defines a family of *symmetric* bilinear forms, and hence for small  $t$  Riemannian metrics, on  $\mathcal{M}$ . The issue here is that the right-hand-side of equation (39) in general does not map symmetric bilinear forms to symmetric bilinear forms, i.e., it is not an operator on the bundle of symmetric bilinear forms on  $\mathcal{M}$ . In fact, the term  $d^{\nabla^t}(\frac{\dot{\mathbf{g}}_t}{\lambda})(U_t, Y, X)$  *a priori* is not symmetric in  $(Y, X)$ <sup>1</sup>. Nevertheless, in Proposition 4.1 we will give a class of examples where the solution  $\mathbf{g}_t$  is symmetric.

Note that, due to the Bianchi identity (8), the bilinear form  $d^{\nabla^t}(\frac{\dot{\mathbf{g}}_t}{\lambda})(U_t, \cdot, \cdot)$  is symmetric, if and only if the 2-form  $d^{\nabla^t}(\frac{\dot{\mathbf{g}}_t}{\lambda})(\cdot, \cdot, U_t)$  vanishes on  $\mathcal{M}$ . Written in terms of the Weingarten operator  $W_t$  of  $(\mathcal{M}, \mathbf{g}_t)$  this is equivalent to the condition that the image of the 2-form  $d^{\nabla^t}W_t$  is orthogonal to  $U_t$  with respect to  $\mathbf{g}_t$ .

#### 4. SOLVING THE EVOLUTION EQUATIONS FOR ANALYTIC DATA

For analytic metrics, we will now overcome the difficulty described in Remark 3.1. In fact, Lemma 3.3 reveals that the problematic evolution equation (39) for  $\mathbf{g}_t$  is equivalent to the curvature condition (26) from Lemma 3.2. In order to overcome the symmetry issue, we next show that for analytic Lorentzian manifolds of the form (5), one can alternatively characterize parallel null vector fields by relaxing (26) and using a method similar to the one in [1]. This approach turns out to yield evolution equations for symmetric bilinear forms  $\mathbf{g}_t$ .

**Lemma 4.1.** *Let  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  be an analytic Lorentzian manifold of the form (5) and let  $V$  be a analytic null vector field on  $\overline{\mathcal{M}}$ . Then  $V$  is parallel for  $\overline{\mathbf{g}}$ , i.e.  $\overline{\nabla}V = 0$ , if and only if the following conditions are satisfied:*

$$\overline{\mathbf{R}}(X, V, V, Y) = 0, \text{ for all } X, Y \in T\mathcal{M}, \quad (43)$$

$$\overline{\nabla}_{\partial_t} \overline{\nabla}_{\partial_t} V = 0, \quad (44)$$

$$\overline{\nabla}_X V|_{\{0\} \times \mathcal{M}} = 0, \text{ for all } X \in T\mathcal{M}|_{\{0\} \times \mathcal{M}}, \quad (45)$$

$$\overline{\nabla}_{\partial_t} V|_{\{0\} \times \mathcal{M}} = 0. \quad (46)$$

*Proof.* Clearly, if  $V$  is parallel, all the conditions follow immediately. On the other hand, assuming the four conditions, it follows from (44) and (46) as in the proof of Lemma 3.2 that  $\overline{\nabla}_{\partial_t} V = 0$  everywhere. It remains to show that  $V$  is also parallel in spacelike directions.

For the rest of the proof let  $X, Y$  be vectors in  $T\mathcal{M}$  or lifts of vector fields on  $\mathcal{M}$  to  $\overline{\mathcal{M}}$ . We consider the sections  $A, B$  of the bundle  $H := (T^\perp)^* \otimes T\overline{\mathcal{M}} \rightarrow \overline{\mathcal{M}}$  and the section  $C$  of the bundle  $K := \Lambda^2(T^\perp)^* \otimes T\overline{\mathcal{M}} \rightarrow \overline{\mathcal{M}}$  defined by

$$\begin{aligned} A(X) &:= \overline{\nabla}_X V, \\ B(X) &:= \overline{\mathbf{R}}(T, X)V, \\ C(X, Y) &:= \overline{\mathbf{R}}(X, Y)V, \end{aligned}$$

for  $X, Y \in T\mathcal{M}$  and denote also by  $\overline{\nabla}$  the covariant derivatives on  $H$  and  $K$  induced by the Levi-Civita connection  $\overline{\nabla}$  of  $\overline{\mathbf{g}}$ . In order to verify that  $A \equiv 0$ , we decompose as in (21),  $V = uT - U$ ,

<sup>1</sup>We thank Olaf Müller for pointing out to us this gap in an earlier draft of this paper.

denote by  $N$  the unit length space-like vector field  $N := \frac{U}{u} \in \Gamma(T^\perp)$  and show that the triplet  $(A, B, C)$  solves the linear PDE

$$\bar{\nabla}_{\partial_t} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = Q \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \quad (47)$$

where  $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \Gamma(H \oplus H \oplus K)$  and  $Q$  is the linear operator on  $H \oplus H \oplus K$  given by

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \mapsto \lambda \begin{pmatrix} \beta + \alpha \circ W \\ d\bar{\nabla}\beta(N, \cdot) + \gamma(W(N), \cdot) + \gamma(N, W(\cdot)) + (\bar{R}(T, \cdot) \wedge \alpha)(N, \cdot) + \gamma(\text{grad}(\log \lambda), \cdot) \\ d\bar{\nabla}\beta + \gamma(W(\cdot), \cdot) + \gamma(\cdot, W(\cdot)) + \bar{R}(T, \cdot) \wedge \alpha \end{pmatrix}.$$

From  $N = T - \frac{1}{u}V$  and  $\bar{\nabla}_{\partial_t}V = 0$  it follows that

$$\bar{\nabla}_T N = \text{grad}(\log \lambda) + \frac{\dot{u}}{\lambda u^2} V. \quad (48)$$

Furthermore, for  $(t, x) \in \{t\} \times \mathcal{M}$  let  $(e_1, \dots, e_n)$  denote a  $\mathbf{g}_t$ -orthonormal basis for  $T_x \mathcal{M}$ . We observe that (43) combined with the skew-symmetries of  $\bar{R}$  yields the following identities (at  $(t, x)$ ):

$$\begin{aligned} \bar{R}(X, V, V, T) &= \bar{R}(X, V, V, N) = 0, \\ \bar{R}(T, X, V, T) &= \bar{R}(T, X, V, N) = \bar{R}(N, X, V, N) = \bar{R}(N, X, V, T), \\ \bar{R}(T, X)V &= \sum_{i=1}^n \bar{R}(T, X, V, e_i)e_i - \bar{R}(T, X, V, T)T \\ &= \sum_{i=1}^n \bar{R}(N, X, V, e_i)e_i - \bar{R}(N, X, V, T)T \\ &= \bar{R}(N, X)V, \end{aligned} \quad (49)$$

$$\bar{R}(T, V)V = \sum_{i=1}^n \bar{R}(N, V, V, e_i)e_i - \bar{R}(N, V, V, N)T = 0, \quad (50)$$

$$\bar{R}(X, V)V = 0. \quad (51)$$

In the following, we view  $\bar{R}$  as 2-form on  $\overline{\mathcal{M}}$  taking values in the  $\bar{\mathbf{g}}$ -skew-symmetric endomorphisms. With these preparations at hand and using the second Bianchi identity for  $\bar{R}$  we compute

$$\begin{aligned} (\bar{\nabla}_T A)(X) &= \bar{\nabla}_T(A(X)) - A(\bar{\nabla}_T X) \\ &= \bar{R}(T, X)V + \bar{\nabla}_{[T, X]}V - \bar{\nabla}_{\bar{\nabla}_T X}V \\ &= B(X) + A(W(X)), \end{aligned}$$

as well as

$$\begin{aligned}
(\bar{\nabla}_T B)(X) &= \bar{\nabla}_T(B(X)) - B(\bar{\nabla}_T X) \\
&= \bar{\nabla}_T(\bar{R}(T, X)V) - \bar{R}(T, \bar{\nabla}_T X)V \\
&\stackrel{(49)}{=} \bar{\nabla}_T(\bar{R}(N, X)V) - \bar{R}(N, \text{pr}_{T^\perp} \bar{\nabla}_T X)V \\
&= \bar{\nabla}_T(\bar{R})(N, X)V + \bar{R}(\bar{\nabla}_T N, X)V + \bar{R}(N, \bar{\nabla}_T X)V - \bar{R}(N, \text{pr}_{T^\perp} \bar{\nabla}_T X)V \\
&\stackrel{(49)}{=} \bar{\nabla}_N(\bar{R})(T, X)V + \bar{\nabla}_X(\bar{R})(N, T)V + \bar{R}(\bar{\nabla}_T N, X)V \\
&\stackrel{(48)}{=} \bar{\nabla}_N(\bar{R}(T, X)V) - \bar{R}(\bar{\nabla}_N T, X)V - \bar{R}(T, \bar{\nabla}_N X)V - \bar{R}(T, X)\bar{\nabla}_N V \\
&\quad + \bar{\nabla}_X(\bar{R}(N, T)V) - \bar{R}(\bar{\nabla}_X N, T)V - \bar{R}(N, \bar{\nabla}_X T)V - \bar{R}(N, T)\bar{\nabla}_X V \\
&\quad + \bar{R}(\text{grad}(\log \lambda), X)V \\
&= (\bar{\nabla}_N B)(X) + C(W(N), X) - \bar{R}(T, X)A(N) \\
&\quad - (\bar{\nabla}_X B)(N) - C(W(X), N) + \bar{R}(T, N)A(X) + C(\text{grad}(\log \lambda), X),
\end{aligned}$$

and

$$\begin{aligned}
(\bar{\nabla}_T C)(X, Y) &= \bar{\nabla}_T(\bar{R}(X, Y)V) - \bar{R}(\bar{\nabla}_T X, Y)V - \bar{R}(X, \bar{\nabla}_T Y)V \\
&= (\bar{\nabla}_T \bar{R})(X, Y)V = (\bar{\nabla}_X \bar{R})(T, Y)V + (\bar{\nabla}_Y \bar{R})(X, T)V \\
&= \bar{\nabla}_X(\bar{R}(T, Y)V) - \bar{R}(\bar{\nabla}_X T, Y)V - \bar{R}(T, \bar{\nabla}_X Y)V - \bar{R}(T, Y)\bar{\nabla}_X V \\
&\quad + (\bar{\nabla}_Y \bar{R})(X, T)V - \bar{R}(\bar{\nabla}_Y X, T)V - \bar{R}(X, \bar{\nabla}_Y T)V - \bar{R}(X, T)\bar{\nabla}_Y V \\
&= (\bar{\nabla}_X B)(Y) + C(W(X), Y) - \bar{R}(T, Y)A(X) - (\bar{\nabla}_Y B)(X) \\
&\quad - C(W(Y), X) + \bar{R}(T, X)A(Y).
\end{aligned}$$

These calculations prove that  $(A, B, C)$  satisfy equation (47), being a linear PDE which separates into  $\partial_t = \lambda \cdot T$ -derivatives on the left-hand-side and spacial derivatives of order at most one on the right-hand-side. At  $t = 0$  we have that  $A = 0$  by assumption. Differentiating this again in direction of  $\mathcal{M}$  and skew-symmetrizing yields that also  $C = 0$  at  $t = 0$ . Finally, it follows that at  $t = 0$  we have  $B(X) = \bar{R}(T, X)V = C(N, X) = 0$  by (49). As all data are analytic, the Cauchy-Kowalevski Theorem guarantees the existence of a unique analytic solution. From the initial conditions it follows that  $A \equiv 0$ ,  $B \equiv 0$ ,  $C \equiv 0$ , i.e.  $\bar{\nabla}_X V = 0$  everywhere.  $\square$

**Remark 4.1.** By inserting  $V = u(T - N)$ , condition (43) from Lemma 4.1 becomes equivalent to

$$\bar{R}(X, T, T, Y) = \bar{R}(N, X, Y, T) + \bar{R}(N, Y, X, T) - \bar{R}(X, N, N, Y) \quad (52)$$

for  $X, Y \in T\mathcal{M}$ . Rewriting (52) in terms of  $t$ -dependent data on  $\mathcal{M}$  using (6)-(12), solving for the  $\dot{\mathbf{g}}_t$ -term and setting  $U_t = u_t N$  yields the equivalent formulation

$$\begin{aligned}
\ddot{\mathbf{g}}_t(X, Y) &= \frac{\lambda^2}{u_t} \left( d^{\nabla^t} \left( \frac{\dot{\mathbf{g}}_t}{\lambda} \right) (U_t, X, Y) + d^{\nabla^t} \left( \frac{\dot{\mathbf{g}}_t}{\lambda} \right) (U_t, Y, X) \right) + \frac{1}{2} \dot{\mathbf{g}}_t(X, \dot{\mathbf{g}}_t^\#(Y)) \\
&\quad + (\log \lambda) \dot{\mathbf{g}}_t(X, Y) + 2\lambda \text{Hess}^t(\lambda)(X, Y) + 2 \frac{\lambda^2}{u_t^2} \text{R}_t(X, U_t, U_t, Y) \\
&\quad + \frac{1}{2u_t^2} \left( \dot{\mathbf{g}}_t(X, Y) \dot{\mathbf{g}}_t(U_t, U_t) - \dot{\mathbf{g}}_t(X, U_t) \dot{\mathbf{g}}_t(Y, U_t) \right).
\end{aligned}$$

The previous calculations directly imply the following statement, which in contrast to Theorem 3.1 we can prove for analytic data only:

**Theorem 4.1.** *Let  $(\mathcal{M}, \mathbf{g}, W, U)$  be an analytic Riemannian manifold together with a field of  $\mathbf{g}$ -symmetric and analytic endomorphisms  $W$ , with corresponding symmetric bilinear form  $\Pi := \mathbf{g}(W., .)$ , and an analytic vector field  $U$  satisfying the following constraint equation*

$$\nabla^{\mathbf{g}} U + uW = 0, \quad (53)$$

where  $u^2 = \mathbf{g}(U, U)$ . Then, for any positive analytic function  $\lambda$  on  $\mathbb{R} \times \mathcal{M}$ , the triple  $(\mathbf{g}_t, U_t, u_t)$  of analytic one-parameter families of Riemannian metrics, vector fields and functions on  $\mathcal{M}$  defines an analytic Lorentzian metric

$$\bar{\mathbf{g}} = -\lambda^2 dt^2 + \mathbf{g}_t$$

on an open neighbourhood  $\bar{\mathcal{U}}(\{0\} \times \mathcal{M}) \subset \mathbb{R} \times \mathcal{M}$  with analytic parallel vector field

$$V = \frac{u_t}{\lambda} \partial_t - U_t,$$

if and only if  $\mathbf{g}_t$ ,  $U_t$  and  $u_t$  satisfy the following system of PDEs on  $\bar{\mathcal{U}}$ ,

$$\begin{aligned} \ddot{\mathbf{g}}_t(X, Y) &= \frac{\lambda^2}{u_t} \left( d^{\nabla^t} \left( \frac{\dot{\mathbf{g}}_t}{\lambda} \right) (U_t, X, Y) + d^{\nabla^t} \left( \frac{\dot{\mathbf{g}}_t}{\lambda} \right) (U_t, Y, X) \right) + \frac{1}{2} \dot{\mathbf{g}}_t(X, \dot{\mathbf{g}}_t^\sharp(Y)) \\ &\quad + (\log \lambda) \dot{\mathbf{g}}_t(X, Y) + 2\lambda \text{Hess}^t(\lambda)(X, Y) + 2\frac{\lambda^2}{u_t^2} R_t(X, U_t, U_t, Y) \\ &\quad + \frac{1}{2u_t^2} \left( \dot{\mathbf{g}}_t(X, Y) \dot{\mathbf{g}}_t(U_t, U_t) - \dot{\mathbf{g}}_t(X, U_t) \dot{\mathbf{g}}_t(Y, U_t) \right). \end{aligned} \quad (54)$$

$$\begin{aligned} \mathbf{g}_t(\ddot{U}_t, X) &= -\frac{\lambda^2}{2u_t} d^{\nabla^t} \left( \frac{\dot{\mathbf{g}}_t}{\lambda} \right) (U_t, X, U_t) - \frac{1}{2} (\log \lambda) \dot{\mathbf{g}}_t(U_t, X) - \lambda \text{Hess}^t(\lambda)(U_t, X) \\ &\quad - \dot{\mathbf{g}}_t(\dot{U}_t, X) + u_t \mathbf{g}_t([\partial_t, \text{grad}^t \lambda], X) + \frac{u}{2} \dot{\mathbf{g}}_t(\text{grad}^t \lambda, X) \\ &\quad + (2\dot{u}_t - d\lambda(U_t)) d\lambda(X), \end{aligned} \quad (55)$$

$$\ddot{u}_t = \mathbf{g}_t([\partial_t, \text{grad}^t \lambda], U_t) + 2d\lambda(\dot{U}_t) + \frac{3}{2} \dot{\mathbf{g}}_t(\text{grad}^t(\lambda), U_t) - u_t \|\text{grad}^t \lambda\|_{\mathbf{g}_t}^2, \quad (56)$$

with the initial conditions

$$\begin{aligned} \mathbf{g}_0 &= \mathbf{g}, \\ \dot{\mathbf{g}}_0 &= -2\lambda_0 \Pi, \\ U_0 &= U, \\ \dot{U}_0 &= u \text{grad}^{\mathbf{g}}(\lambda_0) + \lambda_0 W(U), \\ u_0 &= u, \\ \dot{u}_0 &= d\lambda_0(U). \end{aligned} \quad (57)$$

*Proof.* This is in complete analogy to the proof of Theorem 3.1: one verifies as in Theorem 3.1 and in this case additionally using Remark 4.1 that the equations (53) - (56) and the initial conditions (57) are just a reformulation of the conditions (43)-(46) appearing in Lemma 4.1 in terms of  $t$ -dependent data on  $\mathcal{M}$ . Note that in analogy to Theorem 3.1 the evolution equation (55) arises from substituting the term  $\ddot{\mathbf{g}}_t(U_t, X)$  in (37) via (54) and here additionally using that  $((d \log \lambda) \wedge \dot{\mathbf{g}}_t)(U_t, U_t, X) = 0$ .  $\square$



**Remark 4.2.** In Theorems 3.1 and 4.1 the constraint equation (38) (or (53), respectively) is needed in order to ensure that  $V$  is parallel along  $\mathcal{M}$ . Note also that the constraint  $u^2 - \mathbf{g}(U, U) = 0$  is compatible with the initial conditions (42) (resp. (57)). Indeed,

$$2u\dot{u}_0 = \dot{u}^2|_{t=0} = 2\mathbf{g}_0(\bar{\nabla}_{\partial_t} U|_{t=0}, U_0) = 2\mathbf{g}_0(\dot{U}_0, U_0) - 2\lambda \Pi_0(U_0, U_0) = 2ud\lambda(U).$$

In contrast to (39), the  $\mathbf{g}_t$ -evolution equation (54) is manifestly an equation in the bundle of symmetric bilinear forms on  $\mathcal{M}$ , i.e. at least for small  $t$  the solutions  $\mathbf{g}_t$  are Riemannian metrics on  $\mathcal{M}$ . By the Cauchy-Kowalevski Theorem we obtain the following corollary which gives the statement of Theorem 1 in the introduction:

**Corollary 4.1.** *Let  $(\mathcal{M}, \mathbf{g}, W, U)$  be an analytic Riemannian manifold together with a field of  $\mathbf{g}$ -symmetric, analytic endomorphisms  $W$ , with corresponding symmetric bilinear form  $\Pi := \mathbf{g}(W\cdot, \cdot)$ , and an analytic vector field  $U$  satisfying the following constraint equation*

$$\nabla^{\mathbf{g}} U + uW = 0, \quad (58)$$

where  $u^2 = \mathbf{g}(U, U)$ . Then, for any positive analytic function  $\lambda$  on  $\mathbb{R} \times \mathcal{M}$  there exists an open neighbourhood  $\bar{\mathcal{U}}(\{0\} \times \mathcal{M}) \subset \mathbb{R} \times \mathcal{M}$  and an unique analytic Lorentzian metric

$$\bar{\mathbf{g}} = -\lambda^2 dt^2 + \mathbf{g}_t$$

on  $\bar{\mathcal{U}}$  which admits an analytic null parallel vector field  $V = \frac{u_t}{\lambda} \partial_t - U_t$ , where  $(\mathbf{g}_t, U_t, u_t)$  are solutions of the evolution equations (54), (55), (56) with initial conditions (57).

Finally, we consider an explicit example where we find a solution for the evolutions equations in both Theorems 3.1 and 4.1.

**Proposition 4.1.** *Let  $(M, \mathbf{g}, W, U, u)$  be a Riemannian manifold with an symmetric endomorphism field  $W$ , a vector field  $U$  and a function  $u$  on  $M$  satisfying the constraint equations*

$$\begin{aligned} \nabla^{\mathbf{g}} U &= -uW, \\ \mathbf{g}(U, U) &= u^2 > 0. \end{aligned}$$

Let, in addition,  $W$  be a Codazzi tensor, i.e.,  $d^{\nabla^{\mathbf{g}}} W = 0$ , and  $\lambda = 1$ . Then

$$\mathbf{g}_t := \mathbf{g} - 2t\mathbf{g}(W(\cdot), \cdot) + t^2\mathbf{g}(W^2(\cdot), \cdot) = \mathbf{g}((1 - tW)^2(\cdot), \cdot), \quad (59)$$

$$U(t, x) := \frac{1}{(1 - tW_x)} U(x) = \sum_{k=0}^{\infty} W_x^k(U(x)) t^k, \quad (60)$$

$$u(t, x) := u(x). \quad (61)$$

are solutions to the evolution equations in both Theorem 3.1 and Theorem 4.1. These solutions are defined on

$$\bar{\mathcal{U}}(\{0\} \times \mathcal{M}) := \{(t, x) \in \mathbb{R} \times \mathcal{M} \mid t\|W_x\|_{\mathbf{g}_x} < 1\}.$$

In particular, the above solution  $\mathbf{g}_t$  to the evolution equation (39) is a symmetric bilinear form.

*Proof.* For  $\lambda = 1$ , the evolution equations of Theorem 3.1 reduce to

$$\ddot{\mathbf{g}}_t(X, Y) = \frac{1}{u_t}(d^{\nabla^t} \dot{\mathbf{g}}_t)(U_t, Y, X) - \dot{\mathbf{g}}_t(X, W_t(Y)), \quad (62)$$

$$\begin{aligned} \mathbf{g}(\ddot{U}_t, X) &= -\frac{1}{2u_t}(d^{\nabla^t} \dot{\mathbf{g}}_t)(U_t, X, Y) - \dot{\mathbf{g}}_t(\dot{U}_t, X), \\ \ddot{u}_t &= 0, \end{aligned} \quad (63)$$

whereas the evolution equations of Theorem 4.1 reduce to

$$\begin{aligned} \ddot{\mathbf{g}}_t(X, Y) &= \frac{1}{u_t} \left( (d^{\nabla^t} \dot{\mathbf{g}}_t)(U_t, X, Y) + (d^{\nabla^t} \dot{\mathbf{g}}_t)(U_t, Y, X) \right) - \dot{\mathbf{g}}_t(X, W_t(Y)) \\ &\quad + \frac{2}{u_t^2} R_t(X, U_t, U_t, Y) + \frac{1}{2u_t^2} (\dot{\mathbf{g}}_t(X, Y) \dot{\mathbf{g}}_t(U_t, U_t) - \dot{\mathbf{g}}_t(X, U_t) \dot{\mathbf{g}}_t(Y, U_t)), \end{aligned} \quad (64)$$

$$\begin{aligned} \mathbf{g}(\ddot{U}_t, X) &= -\frac{1}{2u_t} (d^{\nabla^t} \dot{\mathbf{g}}_t)(U_t, X, U_t) - \dot{\mathbf{g}}_t(\dot{U}_t, X) \\ \ddot{u}_t &= 0, \end{aligned} \quad (65)$$

with initial conditions  $\mathbf{g}_0 = \mathbf{g}$ ,  $\dot{\mathbf{g}}_0 = -2\mathbf{II}$ ,  $U_0 = U$ ,  $\dot{U}_0 = W(U)$ ,  $u_0 = u$ ,  $\dot{u}_0 = 0$  in both cases. Hence  $u(t, x) = u(x)$ . It remains to show, that  $\mathbf{g}_t$  and  $U_t$  given by (59) and (60) satisfy (64) and (65) as well as (62) and (63) in case that  $W$  is a Codazzi tensor. To this end, we observe that the definitions (59) and (60) imply

$$\begin{aligned} \dot{\mathbf{g}}_t(X, Y) &= -2\mathbf{g}(W(1 - tW)(X), Y) = -2\mathbf{g}_t\left(\frac{W}{(1-tW)}(X), Y\right) \\ \ddot{\mathbf{g}}_t(X, Y) &= +2\mathbf{g}(W^2(X), Y), \\ \dot{U}_t &= \frac{W}{(1-tW)^2}U, \\ \ddot{U}_t &= \frac{2W^2}{(1-tW)^3}U. \end{aligned} \quad (66)$$

Hence, for the Weingarten operator  $W_t$  of  $(\mathcal{M}, \mathbf{g}_t)$  we obtain

$$W_t = \frac{W}{(1 - tW)}. \quad (67)$$

Next we show, that  $W_t$  is a Codazzi tensor for  $\mathbf{g}_t$ . Since  $-2\mathbf{II}_t = \dot{\mathbf{g}}_t$ , this is equivalent to  $d^{\nabla^t} \dot{\mathbf{g}}_t = 0$ . Since  $W$  is a Codazzi tensor for  $\mathbf{g}$  by definition, the tensor field  $B_t := (1 - tW)$  is Codazzi tensor for  $\mathbf{g}$  as well. Therefore, the Levi-Civita connection of

$$\mathbf{g}_t := B_t^* \mathbf{g} = \mathbf{g}(B_t(\cdot), B_t(\cdot)) = \mathbf{g}(B_t^2(\cdot), \cdot)$$

is given by

$$\nabla^t = B_t^{-1} \circ \nabla^g \circ B_t. \quad (68)$$

It follows

$$\begin{aligned} (d^{\nabla^t} \dot{\mathbf{g}}_t)(X, Y, Z) &= -2X(\mathbf{g}(W(Y), B_t(Z))) + 2\mathbf{g}(B_t W(\nabla_X^t Y), Z) + 2\mathbf{g}(B_t W(Y), \nabla_X^t Z) \\ &\quad + 2Y(\mathbf{g}(W(X), B_t(Z))) - 2\mathbf{g}(B_t W(\nabla_Y^t X), Z) - 2\mathbf{g}(B_t W(X), \nabla_Y^t Z) \\ &= -2\mathbf{g}(\nabla_X^g(W(Y)), B_t(Z)) - 2\mathbf{g}(W(Y), \nabla_X^g(B_t(Z))) \\ &\quad + 2\mathbf{g}(\nabla_Y^g(W(X)), B_t(Z)) + 2\mathbf{g}(W(X), \nabla_Y^g(B_t(Z))) \\ &\quad + 2\mathbf{g}(W(\nabla_X^g(B_t(Y))), Z) + 2\mathbf{g}(W(Y), \nabla_X^g(B_t(Z))) \\ &\quad - 2\mathbf{g}(W(\nabla_Y^g(B_t(X))), Z) - 2\mathbf{g}(W(X), \nabla_Y^g(B_t(Z))) \\ &= -2\mathbf{g}(d^{\nabla^g} W(X, Y), B_t(Z)) - 2\mathbf{g}(W([X, Y], B_t(Z))) \\ &\quad + 2\mathbf{g}(d^{\nabla^g} B_t(X, Y), W(Z)) + 2\mathbf{g}(B_t([X, Y], W(Z))) \\ &= 0. \end{aligned}$$

Finally, we compute the curvature term appearing in (64). It follows from (67) and (68) that

$$\begin{aligned} \nabla^t U_t &= -u W_t, \\ \mathbf{g}_t(U_t, U_t) &= u^2, \end{aligned} \quad (69)$$

and consequently, as  $d^{\nabla^t} W_t = 0$ , we obtain

$$\begin{aligned}
R_t(X, U_t, U_t, Y) &= \mathbf{g}_t(\nabla_X^t(-uW_t(U_t)) - \nabla_{U_t}^t(-uW_t(X)) - \nabla_{[X, U_t]}^t U_t, Y) \\
&= -X(\sqrt{\mathbf{g}_t(U_t, U_t)})\mathbf{g}_t(W_t(U_t), Y) + U_t(\sqrt{\mathbf{g}_t(U_t, U_t)})\mathbf{g}_t(W_t(X), Y) \\
&= \mathbf{g}_t(W_t(X), U_t)\mathbf{g}_t(W_t(Y), U_t) - \mathbf{g}_t(W_t(X), Y)\mathbf{g}_t(W_t(U_t), U_t) \\
&= \frac{1}{4}(\dot{\mathbf{g}}_t(X, U_t)\dot{\mathbf{g}}_t(Y, U_t) - \dot{\mathbf{g}}_t(X, Y)\dot{\mathbf{g}}_t(U_t, U_t)).
\end{aligned} \tag{70}$$

With this property and using equation (67), the evolution equations (62) and (63) as well as (64) and (65) reduce further to the same system, namely

$$\begin{aligned}
\ddot{\mathbf{g}}_t(X, Y) &= -\dot{\mathbf{g}}_t(X, W_t(Y)) = -\dot{\mathbf{g}}_t(X, \frac{W}{(1-tW)}(Y)), \\
\mathbf{g}_t(\ddot{U}_t, X) &= -\dot{\mathbf{g}}_t(\dot{U}_t, X).
\end{aligned} \tag{71}$$

The system (71) obviously has the solution given in equations (66).  $\square$

**Remark 4.3.** Many of the previous statements admit a more general formulation for parallel causal vector fields  $V$  of constant length, i.e.  $\overline{\nabla}V = 0$  and  $\overline{\mathbf{g}}(V, V) \equiv c \leq 0$ . However, in case of a timelike parallel vector field on  $\overline{\mathcal{M}}$  one always has a local metric splitting of  $\overline{\mathcal{M}}$  into a line and a Riemannian factor. For instance, if we replace the constraint equation  $\mathbf{g}(U, U) - u^2 = 0$  in Theorem 3.1 by  $\mathbf{g}(U, U) - u^2 = c = -1$ , then obviously for  $\lambda \equiv 1$  the system of equations has the trivial solution  $\mathbf{g}_t \equiv \mathbf{g}$ ,  $U_t \equiv 0$  and  $u_t \equiv 1$  which gives the parallel timelike vector field  $\partial_t$  on the product metric  $\overline{\mathbf{g}} = -dt^2 + \mathbf{g}$ .

## 5. CONSTRAINT AND EVOLUTION EQUATIONS FOR PARALLEL NULL SPINORS

In this section we assume in addition, that  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  is a Lorentzian *spin* manifold. For a spinor field  $\phi$  on  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  we define its Dirac current  $V_\phi \in \mathfrak{X}(\overline{\mathcal{M}})$  by

$$\overline{\mathbf{g}}(V_\phi, X) = -\langle X \cdot \phi, \phi \rangle, \quad \forall X \in \mathfrak{X}(\overline{\mathcal{M}}).$$

The vector field  $V_\phi$  is future oriented, causal, i.e.,  $\overline{\mathbf{g}}(V_\phi, V_\phi) \leq 0$  and the zero sets of  $V_\phi$  and  $\phi$  coincide. If  $\phi$  is parallel,  $V_\phi$  is parallel as well, and thus either null or time-like. We call a spinor field  $\phi$  *null*, if its Dirac current  $V_\phi$  is null. In this case we have  $V_\phi \cdot \phi = 0$  and  $\langle \phi, \phi \rangle = 0$ .

From now on, we assume that  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  admits a *parallel null* spinor field  $\phi$ . Then, for its Dirac current  $V := V_\phi$  we apply the notations and results of Section 3. We fix a time-orientation  $T$ , and consider as in (20) and (21) the projection  $U$  of  $-V$  onto  $T\mathcal{M}$ , the function  $u := -\overline{\mathbf{g}}(V, T)$  and the unit vector field  $N := \frac{1}{u}U$ . Since  $\phi$  is parallel, the Ricci endomorphism is zero or 2-step nilpotent, i.e.,  $\overline{\text{Ric}}^2 = 0$ . This is equivalent to  $\overline{\text{Ric}} = f \cdot (V^\flat)^2$  with a function  $f$  on  $\overline{\mathcal{M}}$ . This implies

$$\begin{aligned}
\overline{\text{Ric}}(T, T) &= f u^2 \\
\overline{\text{Ric}}(X, T) &= f u^2 \mathbf{g}(N, X) \\
\overline{\text{Ric}}(X, Y) &= f u^2 \mathbf{g}(N, X)\mathbf{g}(N, Y)
\end{aligned}$$

for  $X, Y \in T\mathcal{M}$ . Therefore,

$$\begin{aligned}
\overline{\text{Ric}}(T, T) &= \overline{\text{Ric}}(N, N) = \overline{\text{Ric}}(N, T) = f u^2, \\
\overline{\text{Ric}}(X, Y) &= 0 \quad \text{if } X \in \text{span}(T, N)^\perp \text{ or } Y \in \text{span}(T, N)^\perp.
\end{aligned}$$

In particular, the scalar curvature  $\overline{\text{scal}}$  of  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  vanishes. If  $(\mathcal{M}, \mathbf{g})$  is a space-like hypersurface of  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  with the normal vector field  $T$ , the second fundamental form  $\text{II}$ , and the Weingarten operator  $W$ , the Ricci-tensor and the scalar curvature of  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  and  $(\mathcal{M}, \mathbf{g})$  are related by

$$\begin{aligned}\overline{\text{Ric}}(T, T) &= \overline{\text{Ric}}(N, T) = \overline{\text{Ric}}(N, N) = \text{tr}_{\mathbf{g}} d^{\nabla} \text{II}(N) \\ \overline{\text{Ric}}(X, T) &= \text{tr}_{\mathbf{g}} d^{\nabla} \text{II}(X) \\ \overline{\text{Ric}}(X, Y) &= \text{Ric}(X, Y) - d^{\nabla} \text{II}(N, X, Y) - \text{II}(X, W(Y)) + \text{tr}_{\mathbf{g}} \text{II} \cdot \text{II}(X, Y) \\ \overline{\text{scal}} &= \text{scal} - 2\text{tr}_{\mathbf{g}} d^{\nabla} \text{II}(N) - \|\text{II}\|_{\mathbf{g}}^2 + (\text{tr}_{\mathbf{g}} \text{II})^2.\end{aligned}$$

for all vectors  $X, Y \in T\mathcal{M}$ , where  $d^{\nabla} \text{II}(X) := d^{\nabla} \text{II}(X, \cdot, \cdot)$ . Hence, we obtain

**Proposition 5.1** (Ricci constraint conditions).

Let  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  be a Lorentzian spin manifold with a parallel null spinor field, and let  $(\mathcal{M}, \mathbf{g})$  be a space-like hypersurface of  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  with normal vector field  $T$ , second fundamental form  $\text{II}$  and Weingarten operator  $W$ . Then for all vectors  $X, Y \in N^{\perp} \subset T\mathcal{M}$ ,

$$\begin{aligned}\text{tr}_{\mathbf{g}} d^{\nabla} \text{II}(X) &= 0, \\ \text{tr}_{\mathbf{g}} d^{\nabla} \text{II}(N) \cdot N &= \text{div}_{\mathbf{g}} \text{II} + \text{grad}_{\mathbf{g}}(\text{tr}_{\mathbf{g}} \text{II}), \\ 2\text{tr}_{\mathbf{g}} d^{\nabla} \text{II}(N) &= \text{scal} - \|\text{II}\|_{\mathbf{g}}^2 + (\text{tr}_{\mathbf{g}} \text{II})^2,\end{aligned}$$

and

$$\begin{aligned}\text{Ric}(X, Y) &= d^{\nabla} \text{II}(N, X, Y) + \text{II}(W(X), Y) - \text{tr}_{\mathbf{g}} \text{II} \cdot \text{II}(X, Y), \\ \text{Ric}(X, N) &= \text{II}(W(X), N) - \text{tr}_{\mathbf{g}}(\text{II}) \cdot \text{II}(X, N), \\ \text{Ric}(N, N) &= \text{tr}_{\mathbf{g}} d^{\nabla} \text{II}(N) - \text{tr}_{\mathbf{g}}(\text{II}) \cdot \text{II}(N, N) + \text{II}(W(N), N).\end{aligned}$$

Let  $(\overline{S}, \nabla^{\overline{S}})$  denote the spinor bundle of  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  with the covariant derivative induced by the Levi-Civita connection, and let  $(S, \nabla^S)$  be the spinor bundle of the space-like hypersurface  $(\mathcal{M}, \mathbf{g})$  with its spin derivative. Then there is a canonical identification of  $S$  with  $\overline{S}|_{\mathcal{M}}$  if  $n$  is even and of  $S$  with the half-spinors  $\overline{S}|_{\mathcal{M}}^+$  if  $n$  is odd. In this identification, the Clifford product with a vector field  $X$  on  $\mathcal{M}$  in both bundles is related via

$$X \cdot \varphi = i T^{\perp} X^{\perp} \phi|_{\mathcal{M}}, \quad (72)$$

where  $\varphi \in \Gamma(S)$  is identified with  $\phi|_{\mathcal{M}} \in \Gamma(\overline{S}|_{\mathcal{M}}^{(+)})$ . In the following we will omit the  $^{\perp}$  over the Clifford multiplication in  $\overline{S}$  in order to keep the notation simple, it will always be clear in which spinor bundle we are working. The Dirac current  $U_{\psi}$  of a spinor field  $\psi$  on a Riemannian spin manifold  $(\mathcal{M}, \mathbf{g})$  is given by

$$\mathbf{g}(U_{\psi}, X) := -i(X \cdot \psi, \psi), \quad X \in T\mathcal{M}. \quad (73)$$

If  $\phi \in \Gamma(\overline{S}|_{\mathcal{M}}^{(+)})$  is a spinor field on  $\overline{\mathcal{M}}$  and  $\varphi := \phi|_{\mathcal{M}} \in \Gamma(S)$  its restriction to  $\mathcal{M}$ , the Dirac currents satisfies

$$(V_{\phi})|_{\mathcal{M}} = \|\varphi\|^2 T|_{\mathcal{M}} - U_{\varphi}.$$

Using the above identification of the spinor bundles, the conditions  $\nabla^{\overline{S}} \phi = 0$  and  $V_{\phi} \cdot \phi = 0$  translate into the following conditions for the spinor field  $\varphi = \phi|_{\mathcal{M}}$ :

**Proposition 5.2** (Spin constraint conditions).

If  $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$  is a spin manifold with a parallel null spinor field  $\phi$ . Then the spinor field  $\varphi := \phi|_{\mathcal{M}}$  on the space-like hypersurface  $(\mathcal{M}, \mathbf{g})$  satisfies

$$\nabla_X^S \varphi = \frac{i}{2} W(X) \cdot \varphi \quad \forall X \in T\mathcal{M}, \quad (74)$$

$$U_\varphi \cdot \varphi = i u_\varphi \varphi, \quad (75)$$

where  $W$  is the Weingarten operator of  $(\mathcal{M}, \mathbf{g})$  and  $u_\varphi = \sqrt{\mathbf{g}(U_\varphi, U_\varphi)} = \|\varphi\|^2$ .

For a detailed explanation of the identifications used above and a proof of Proposition 5.2 we refer to [2] and [6]. For an arbitrary symmetric  $(1,1)$ -tensor field  $W$  on a Riemannian spin manifold  $(\mathcal{M}, \mathbf{g})$  we call a spinor field  $\varphi$  on  $\mathcal{M}$ , satisfying (74) and (75), an *imaginary W-Killing spinor*. In [2] Bär, Gauduchon and Moroianu consider the case of a *real W-Killing spinor*  $\varphi$  on a semi-Riemannian manifold  $(\mathcal{M}, \mathbf{g})$ . They show that, if  $W$  is a Codazzi tensor, the manifold  $(\mathcal{M}, \mathbf{g})$  can be embedded as a hypersurface into a Ricci flat manifold  $(\overline{\mathcal{M}}, \overline{\mathbf{g}} = dr^2 + \mathbf{g}_r)$  equipped with a parallel spinor which restricts to  $\varphi$ .

We will now show, that any imaginary W-Killing spinor  $\varphi$  on a Riemannian spin manifold  $(\mathcal{M}, \mathbf{g})$  can be extended to a parallel light-like spinor field  $\phi$  on an open neighbourhood  $\overline{\mathcal{U}}(\{0\} \times \mathcal{M}) \subset \mathbb{R} \times \mathcal{M}$  with a Lorentzian metric  $\overline{\mathbf{g}} := -\lambda^2 dt^2 + \mathbf{g}_t$  such that  $\phi|_{\mathcal{M}} = \varphi$ , at least if all given data are real analytic. First we show, that the Dirac current  $U_\varphi$  of  $\varphi$  satisfies the constraint conditions (53) of Theorem 4.1.

**Lemma 5.1.** *Let  $\varphi$  be an imaginary W-Killing spinor on  $(\mathcal{M}, \mathbf{g})$ . Then the Dirac current  $U_\varphi$  of  $\varphi$  satisfies*

$$X(u_\varphi) = -\mathbf{g}(W(X), U_\varphi), \quad (76)$$

$$\nabla_X U_\varphi = -u_\varphi W(X) \quad (77)$$

for all vector fields  $X$  on  $\mathcal{M}$ .

*Proof.* We write  $u := u_\varphi$  and  $U := U_\varphi$ . Since  $u = (\varphi, \varphi)$ , we obtain

$$\begin{aligned} X(u) &= (\nabla_X^S \varphi, \varphi) + \overline{(\nabla_X^S \varphi, \varphi)} \\ &= \frac{i}{2} (W(X) \cdot \varphi, \varphi) + \overline{\frac{i}{2} (W(X) \cdot \varphi, \varphi)} \\ &= -\mathbf{g}(W(X), U), \end{aligned}$$

by equation (73). Differentiating (75) and inserting (74) and (76) gives

$$\begin{aligned} \nabla_X^S (U \cdot \varphi) &= \nabla_X U \cdot \varphi + U \cdot \nabla_X^S \varphi \\ &= \nabla_X U \cdot \varphi + \frac{i}{2} U \cdot W(X) \cdot \varphi \\ &= \nabla_X U \cdot \varphi - \frac{i}{2} W(X) \cdot U \cdot \varphi - i \mathbf{g}(U, W(X)) \varphi \\ &= \nabla_X U \cdot \varphi + \frac{1}{2} u W(X) \cdot \varphi - i \mathbf{g}(U, W(X)) \varphi, \\ \nabla_X^S (i u \varphi) &= i X(u) \varphi + i u \nabla_X^S \varphi \\ &= -i \mathbf{g}(W(X), U) \varphi - \frac{1}{2} u W(X) \cdot \varphi. \end{aligned}$$

Hence,  $(\nabla_X U + u W(X)) \cdot \varphi = 0$ , which shows (77).  $\square$

Now, let us suppose, that the Riemannian spin manifold  $(\mathcal{M}, \mathbf{g})$  and the field of  $\mathbf{g}$ -symmetric endomorphisms  $W$  are real analytic. Then the Dirac current  $U_\varphi$  of an imaginary W-Killing spinor  $\varphi$  and its length  $u_\varphi$  are real analytic as well and satisfy the constraint equations of Theorem 4.1 and Corollary 4.1. Hence, for any positive analytic function  $\lambda$  on  $\mathbb{R} \times \mathcal{M}$  there

exists an open neighbourhood  $\overline{U}$  of  $\mathcal{M} \simeq \{0\} \times \mathcal{M} \subset \mathbb{R} \times \mathcal{M}$  and an unique analytic Lorentzian metric  $\overline{\mathbf{g}} = -\lambda^2 dt^2 + \mathbf{g}_t$  on  $\overline{U}$ , which admits an analytic parallel null vector field  $V = \frac{u_t}{\lambda} \partial_t - U_t$ , where  $(\mathbf{g}_t, U_t, u_t)$  are solutions of the evolution equations (54, 55, 56) with initial conditions (57), where  $U := U_\varphi$  and  $u := u_\varphi$ . Now we observe the following property of the parallel null vector field  $V = \frac{u_t}{\lambda} \partial_t - U_t$ .

**Lemma 5.2.** *Let  $\phi \in \Gamma(\overline{S}^{(+)})$  be defined by parallel transport of  $\varphi \in \Gamma(S \simeq \overline{S}_{|\mathcal{M}}^{(+)})$  along the  $t$ -lines  $\gamma_x(t) = (t, x)$  of  $\overline{U}$ . Then  $V$  is the Dirac current of  $\phi$ .*

*Proof.* We use the global vector fields  $T$  and  $N := u^{-1}U$  to reduce the frame bundle of  $(\overline{U}, \overline{\mathbf{g}})$  to the subgroup  $\mathbf{SO}(n-1) \subset \mathbf{SO}(1, n)$ . Then, the spin structure of  $(\overline{U}, \overline{\mathbf{g}})$  is given by a spin structure  $\widehat{Q}$  of the reduced frame bundle  $\widehat{P}$ . Let  $\widehat{S} := \widehat{Q} \times_{Spin(n-1)} \Delta_{n-1}$ . Since the spinor modul  $\Delta_{1,n}$  is isomorphic to  $\Delta_{n-1} \otimes \Delta_{1,1}$ , we can identify the spinor bundle  $\overline{S}$  of  $(\overline{U}, \overline{\mathbf{g}})$  with  $\widehat{S} \otimes \Delta_{1,1}$ , where  $T, N$  and  $X \in \text{span}(T, N)^\perp$  act on  $\widehat{\psi} \otimes u(\varepsilon) \in \widehat{S} \otimes \Delta_{1,1}$  by

$$\begin{aligned} T \cdot (\widehat{\psi} \otimes u(\varepsilon)) &= -\widehat{\psi} \otimes u(-\varepsilon), \\ N \cdot (\widehat{\psi} \otimes u(\varepsilon)) &= \varepsilon \widehat{\psi} \otimes u(-\varepsilon), \\ X \cdot (\widehat{\psi} \otimes u(\varepsilon)) &= -\varepsilon (X \cdot \widehat{\psi}) \otimes u(\varepsilon). \end{aligned}$$

Here,  $\{u(\varepsilon) := \begin{pmatrix} 1 \\ -\varepsilon i \end{pmatrix} \mid \varepsilon = \pm 1\}$  denotes an unitary basis in the complex vector space  $\Delta_{1,1} \simeq \mathbb{C}^2$ . Then for  $\psi = \widehat{\psi}_1 \otimes u(1) + \widehat{\psi}_{-1} \otimes u(-1) \in \Gamma(\overline{S})$  it holds:

$$V \cdot \psi = 0 \iff T \cdot \psi = N \cdot \psi \iff \psi = T \cdot N \cdot \psi \iff \psi = \widehat{\psi}_{-1} \otimes u(-1). \quad (78)$$

Now, since  $U \cdot \varphi = iu\varphi$  in  $\Gamma(S)$ , the spinor field  $\phi_{|\mathcal{M}} \in \Gamma(\overline{S}_{|\mathcal{M}}^{(+)})$  satisfies  $iT \cdot U \cdot \phi_{|\mathcal{M}} = iu\phi_{|\mathcal{M}}$  (see (72)) and therefore,  $T \cdot N \cdot \phi_{|\mathcal{M}} = \phi_{|\mathcal{M}}$ . This shows, that  $(V \cdot \phi)_{|\mathcal{M}} = 0$ . But  $V$  as well as  $\phi$  are parallel along the  $t$ -lines and we obtain

$$\nabla_{\partial_t}^{\overline{S}}(V \cdot \phi) = \overline{\nabla}_{\partial_t} V \cdot \phi + V \cdot \nabla_{\partial_t}^{\overline{S}} \phi = 0.$$

Thus the spinor field  $V \cdot \phi$  is parallel along the  $t$ -lines as well. Since it vanishes on  $\{0\} \times \mathcal{M}$ , it vanishes on  $\overline{U}$ . Then for the Dirac current of  $\phi$  hold

$$\begin{aligned} \overline{\mathbf{g}}(V_\phi, T) &= -\langle T \cdot \phi, \phi \rangle = -(T \cdot T \cdot \phi, \phi) = -(\phi, \phi), \\ \overline{\mathbf{g}}(V_\phi, N) &= -\langle N \cdot \phi, \phi \rangle = -(T \cdot N \cdot \phi, \phi) = -(\phi, \phi), \\ \overline{\mathbf{g}}(V_\phi, X) &= -\langle X \cdot \phi, \phi \rangle = -(T \cdot X \cdot \phi, \phi) = ((X \cdot \widehat{\phi}_{-1} \otimes u(1), \widehat{\phi}_{-1} \otimes u(-1)) = 0 \end{aligned}$$

for  $X \in \text{span}(T, N)^\perp$ . This shows, that  $V = \frac{u}{\|\phi\|^2} V_\phi$ . Since  $V$  and  $V_\phi$  are parallel along the  $t$ -lines,  $u(t, x) = c(x)\|\phi(t, x)\|^2$ . Because of  $u(0, x) = \|\varphi(x)\|^2 = \|\phi(0, x)\|^2$  we have  $c(x) = 1$ . This shows that  $V$  is the Dirac current of  $\phi$ .  $\square$

Using a similar method as the authors of [1], we will now show that  $\phi$  is parallel on  $(\overline{U}, \overline{\mathbf{g}})$ .

**Theorem 5.1.** *Let  $(\mathcal{M}, \mathbf{g})$  be an analytic Riemannian spin manifold with an analytic  $\mathbf{g}$ -symmetric endomorphism field  $W$  and  $\varphi$  an imaginary  $W$ -Killing spinor on  $(\mathcal{M}, \mathbf{g})$ , and let*

$$(\overline{U}(\{0\} \times \mathcal{M}), \overline{\mathbf{g}} = -\lambda^2 dt^2 + \mathbf{g}_t)$$

*be the Lorentzian manifold with parallel null vector field  $V$ , arising as the solutions of the evolution equations (54)-(56) in Corollary 4.1 with the initial conditions (57) given by  $(\mathcal{M}, \mathbf{g}, W, U_\varphi)$ . Let  $\phi$  be the spinor field on  $(\overline{U}, \overline{\mathbf{g}})$  obtained by parallel transport of  $\varphi$  along the  $t$ -lines  $t \mapsto (t, x)$ . Then  $\phi$  is a parallel spinor field on  $(\overline{U}, \overline{\mathbf{g}})$  with the Dirac current  $V_\phi = V$ .*

*Proof.* Since  $\nabla_{\partial_t}^{\bar{S}}\phi = 0$  by definition, it remains to show, that  $\nabla_X^{\bar{S}}\phi = 0$  for all vector fields  $X$  on  $\bar{\mathcal{U}}$  tangent to  $\mathcal{M}$ . In the following we will consider the bundle  $\Lambda^k T^* \mathcal{M} \otimes \bar{S}$  of  $k$ -forms on  $T\mathcal{M}$  with values in  $\bar{S}$  with the covariant derivative  $\bar{\nabla}$  induced by the Levi-Civita connection of  $\bar{g}$  and the spin connection  $\nabla^{\bar{S}}$ :

$$\bar{\nabla}_X(\omega)(Y_1, \dots, Y_k) := \nabla_X^{\bar{S}}(\omega(Y_1, \dots, Y_k)) - \sum_{i=1}^k \omega(Y_1, \dots, Y_{i-1}, \text{pr}_{T^\perp} \bar{\nabla}_X Y_i, Y_{i+1}, \dots, Y_k)$$

for  $\omega \in \Gamma(\Lambda^k T^* \mathcal{M} \otimes \bar{S})$ .

We consider now the section  $\begin{pmatrix} A \\ B \end{pmatrix}$  of the bundle  $E := (T^* \mathcal{M} \otimes \bar{S}) \oplus (\Lambda^2 T^* \mathcal{M} \otimes \bar{S})$  over  $\bar{\mathcal{U}}$ , defined by

$$\begin{aligned} A(X) &:= \nabla_X^{\bar{S}} \phi, \\ B(X, Y) &:= R^{\bar{S}}(X, Y) \phi \end{aligned}$$

for  $X, Y \in T\mathcal{M}$ . In order to show that  $A = 0$ , we show that  $\begin{pmatrix} A \\ B \end{pmatrix}$  satisfies the following linear PDE on  $E$

$$\bar{\nabla}_{\partial_t} \begin{pmatrix} A \\ B \end{pmatrix} = Q \begin{pmatrix} A \\ B \end{pmatrix}, \quad (79)$$

where  $Q$  is the linear operator on  $E$ , given by

$$Q \begin{pmatrix} \omega \\ \mu \end{pmatrix} = \lambda \begin{pmatrix} +\mu(N, \cdot) + \omega \circ W \\ -d\bar{\nabla}\mu(\cdot, N) + d(\ln \frac{\lambda}{u}) \wedge \mu(N, \cdot) + \frac{1}{2}\bar{R}(N, \cdot) \wedge \omega \end{pmatrix}.$$

To this aim, we use the following formula for the curvature of the spin connection in  $\bar{S}$  (see [5]):

$$R^{\bar{S}}(X, Y)\psi = \frac{1}{2}\bar{R}(X, Y) \cdot \psi,$$

where on the right hand side stands the Clifford multiplication of  $\psi$  with the 2-form

$$\bar{R}(X, Y) = \sum_{i < j} \bar{R}(X, Y, \cdot, \cdot).$$

Since  $\bar{\nabla}V = 0$  and  $V = u(T - N)$ , the curvature of  $\bar{g}$  satisfies  $\bar{R}(T, X, Y, Z) = \bar{R}(N, X, Y, Z)$  for all vectors  $X, Y, Z \in T\bar{\mathcal{U}}$ . Hence,

$$R^{\bar{S}}(T, X)\psi = R^{\bar{S}}(N, X)\psi \quad \forall X \in T\mathcal{M}. \quad (80)$$

Furthermore we obtain

$$\bar{\nabla}_X N = X(\ln u)T - X(\ln u)N - W(X) \quad \forall X \in T\mathcal{M}. \quad (81)$$

Moreover,

$$\begin{aligned} B(\text{pr}_{T^\perp} \bar{\nabla}_T X, Y) &= R^{\bar{S}}(\bar{\nabla}_T X, Y)\phi - X(\ln \lambda)R^{\bar{S}}(T, Y)\phi \\ &= R^{\bar{S}}(\bar{\nabla}_T X, Y)\phi - X(\ln \lambda)B(N, Y) \end{aligned} \quad (82)$$

for all vector fields  $X, Y \in \Gamma(T^\perp)$ , since  $\bar{\nabla}_T X = X(\ln \lambda)T + \text{pr}_{T^\perp} \bar{\nabla}_T X$ .

Then, using (80), (81), (82) and the second Bianchi identity for  $\bar{R}$ , we obtain for all vector fields



$X, Y \in \Gamma(T\mathcal{M})$

$$\begin{aligned}
(\overline{\nabla}_T A)(X) &= \nabla_T^{\overline{S}}(A(X)) - A(\text{pr}_{T^\perp} \overline{\nabla}_T X) \\
&= \nabla_T^{\overline{S}} \nabla_X^{\overline{S}} \phi - \nabla_{\overline{\nabla}_T X}^{\overline{S}} \phi \\
&= R^{\overline{S}}(T, X) \phi + \nabla_X^{\overline{S}} \nabla_T^{\overline{S}} \phi + \nabla_{[T, X]}^{\overline{S}} \phi - \nabla_{\overline{\nabla}_T X}^{\overline{S}} \phi \\
&= R^{\overline{S}}(N, X) \phi - \nabla_{\overline{\nabla}_X T}^{\overline{S}} \phi \\
&= B(N, X) + A(W(X))
\end{aligned}$$

and

$$\begin{aligned}
(\overline{\nabla}_T B)(X, Y) &= \nabla_T^{\overline{S}}(R^{\overline{S}}(X, Y) \phi) - B(\text{pr}_{T^\perp} \overline{\nabla}_T X, Y) - B(X, \text{pr}_{T^\perp} \overline{\nabla}_T Y) \\
&= \frac{1}{2} \overline{\nabla}_T(\overline{R}(X, Y)) \cdot \phi - \frac{1}{2} \overline{R}(\overline{\nabla}_T X, Y) \cdot \phi - \frac{1}{2} \overline{R}(X, \overline{\nabla}_T Y) \cdot \phi \\
&\quad + X(\ln \lambda) B(N, Y) - Y(\ln \lambda) B(N, X) \\
&= \frac{1}{2} (\overline{\nabla}_T \overline{R})(X, Y) \cdot \phi + X(\ln \lambda) B(N, Y) - Y(\ln \lambda) B(N, X) \\
&= +\frac{1}{2} (\overline{\nabla}_X \overline{R})(T, Y) \cdot \phi - \frac{1}{2} (\overline{\nabla}_Y \overline{R})(T, X) \cdot \phi + X(\ln \lambda) B(N, Y) - Y(\ln \lambda) B(N, X) \\
&= \nabla_X^{\overline{S}}(R^{\overline{S}}(T, Y) \phi) - \frac{1}{2} \overline{R}(T, Y) \cdot \nabla_X^{\overline{S}} \phi - R^{\overline{S}}(\overline{\nabla}_X T, Y) \phi - R^{\overline{S}}(N, \text{pr}_{T^\perp} \overline{\nabla}_X Y) \phi \\
&\quad - \nabla_Y^{\overline{S}}(R^{\overline{S}}(T, X) \phi) + \frac{1}{2} \overline{R}(T, X) \cdot \nabla_Y^{\overline{S}} \phi + R^{\overline{S}}(\overline{\nabla}_Y T, X) \phi + R^{\overline{S}}(N, \text{pr}_{T^\perp} \overline{\nabla}_Y X) \phi \\
&\quad + X(\ln \lambda) B(N, Y) - Y(\ln \lambda) B(N, X) \\
&= (\overline{\nabla}_X B)(N, Y) - \frac{1}{2} \overline{R}(N, Y) A(X) + B(W(X), Y) + B(\text{pr}_{T^\perp} \overline{\nabla}_X N, Y) \\
&\quad - (\overline{\nabla}_Y B)(N, X) + \frac{1}{2} \overline{R}(N, X) A(Y) - B(W(Y), X) + B(\text{pr}_{T^\perp} \overline{\nabla}_Y N, X) \\
&\quad + X(\ln \lambda) B(N, Y) - Y(\ln \lambda) B(N, X) \\
&= (\overline{\nabla}_X B)(N, Y) - \frac{1}{2} \overline{R}(N, Y) A(X) - X(\ln u) B(N, Y) \\
&\quad - (\overline{\nabla}_Y B)(N, X) + \frac{1}{2} \overline{R}(N, X) A(Y) + Y(\ln u) B(N, X) \\
&\quad + X(\ln \lambda) B(N, Y) - Y(\ln \lambda) B(N, X) \\
&= (\overline{\nabla}_X B)(N, Y) - (\overline{\nabla}_Y B)(N, X) + X(\ln \frac{\lambda}{u}) B(N, Y) - Y(\ln \frac{\lambda}{u}) B(N, X) \\
&\quad - \frac{1}{2} \overline{R}(N, Y) \cdot A(X) + \frac{1}{2} \overline{R}(N, X) \cdot A(Y).
\end{aligned}$$

This shows, that the section  $\begin{pmatrix} A \\ B \end{pmatrix}$  in  $E$  solves the linear PDE (79). Now, let us consider the section  $\begin{pmatrix} A \\ B \end{pmatrix}$  on the initial hypersurface  $\{0\} \times \mathcal{M} \simeq \mathcal{M}$ . First, we have

$$A(X)|_{t=0} = \overline{\nabla}_X \phi|_{t=0} = 0, \quad (83)$$

for all  $X \in T\mathcal{M}$ , since  $\phi$  is defined by parallel transport of the imaginary Killing spinor  $\varphi$  and (83) is the  $\overline{S}|_{\mathcal{M}}$  correspondence for the Killing condition (74) of  $\varphi \in \Gamma(S)$ . Then of course,  $B(X, Y)|_{t=0} = R^{\overline{S}}(X, Y) \phi|_{t=0} = 0$  for all  $X, Y \in T\mathcal{M}$ .

To summarize, the section  $\begin{pmatrix} A \\ B \end{pmatrix} \in \Gamma(E)$  solves the linear PDE (79), where the time derivative  $\overline{\nabla}_{\partial_t}$  separates on the left hand side and on the right hand side are only derivatives in the space-direction of maximally first order, with initial condition  $\begin{pmatrix} A \\ B \end{pmatrix}|_{t=0} = 0$  on the initial space-like hypersurface  $\mathcal{M} \simeq \{0\} \times \mathcal{M}$ . Since all data are real analytic, the Cauchy-Kowalevski Theorem guaranties an unique analytic solution. Hence, by the initial conditions, this solution is identically zero. This shows that the spinor field  $\phi$  on  $(\overline{\mathcal{U}}, \overline{\mathbf{g}})$  is parallel.  $\square$

**Remark 5.1.** Our method needs analyticity of all data. In the Riemannian analogue, smooth real W-Killing spinors in general cannot be extended to parallel spinors (see [9] and [1]). Having the analogous situation for the Einstein equation in mind, where the work of Choquet-Bruhat [11] dealt with the smooth case, the question remains whether in the Lorentzian setting smoothness is sufficient for extending W-Killing spinors to parallel ones, or if there are smooth examples that cannot be extended.

## 6. RIEMANNIAN MANIFOLDS SATISFYING THE CONSTRAINT EQUATIONS

In this section we describe some examples of *complete* Riemannian manifolds  $(\mathcal{M}, \mathbf{g}, W, U)$  satisfying the constraint conditions (53) of Theorem 4.1 and admitting imaginary W-Killing spinors. First recall that for a vector field  $U$  the endomorphism field  $\nabla U$  is symmetric with respect to  $\mathbf{g}$  if and only if the metric dual  $U^\flat = \mathbf{g}(U, \cdot)$  is a closed one form. This implies that a Riemannian manifold  $(\mathcal{M}, \mathbf{g}, W, U)$  satisfying the constraint conditions (53) of Theorem 4.1 is foliated in integral manifolds  $\mathcal{F}$  of  $U^\perp = \text{Ker}(U^\flat)$  with the second fundamental form  $\Pi^\mathcal{F}(X, Y) = \Pi(X, Y)$  for all  $X, Y \in U^\perp$  and the Weingarten operator  $W^\mathcal{F} := \text{pr}_{U^\perp} \circ W$ .

Now we collect some integrability conditions for an imaginary W-Killing spinor.

**Lemma 6.1.** *Let  $(\mathcal{M}, \mathbf{g})$  be a Riemannian spin manifold with a symmetric endomorphism field  $W$  and suppose that there is a (non-trivial) imaginary W-Killing spinor  $\varphi$  as in (74) and (75) with Dirac current  $U_\varphi$  of  $\varphi$  and  $u_\varphi = \|\varphi\|^2 = \sqrt{\mathbf{g}(U_\varphi, U_\varphi)} > 0$ . Then  $U_\varphi^\flat$  is a closed 1-form and the integral manifolds of the distribution  $U_\varphi^\perp$  are Riemannian spin manifolds with a parallel spinor field. Moreover,  $U_\varphi$  and  $u_\varphi$  satisfy*

$$\begin{aligned} \nabla_X U_\varphi &= -u_\varphi W(X), \\ W(U_\varphi) &= \text{grad } u_\varphi, \\ \mathbf{g}(d^\nabla W(X, Y), U_\varphi) &= 0, \\ X(u_\varphi) &= -\mathbf{g}(W(X), U_\varphi), \\ YX(u_\varphi) &= -\mathbf{g}((\nabla_Y W)(X) + W(\nabla_Y X), U_\varphi) + u_\varphi \mathbf{g}(W(X), W(Y)), \\ \text{Hess}(u_\varphi)(X, Y) &= -\mathbf{g}((\nabla_X W)(Y), U_\varphi) + u_\varphi \mathbf{g}(W(X), W(Y)). \end{aligned}$$

for  $X, Y \in T\mathcal{M}$ . The Dirac operator  $D$  and the Bochner-Laplace operator  $\nabla^* \nabla$  on the spinor bundle applied to  $\varphi$  yield

$$\begin{aligned} D\varphi &= -\frac{i}{2} \text{tr}_{\mathbf{g}}(W)\varphi, \\ D^2\varphi &= -\frac{1}{4}(\text{tr}_{\mathbf{g}}(W))^2\varphi - \frac{i}{2} \text{grad}(\text{tr}_{\mathbf{g}}(W)) \cdot \varphi, \\ \nabla^* \nabla \varphi &= \frac{i}{2} \text{div}_{\mathbf{g}}(W) \cdot \varphi - \frac{1}{4} \text{tr}_{\mathbf{g}}(W^2)\varphi. \end{aligned}$$

Furthermore, the curvature of the spin connection satisfies

$$\begin{aligned} R^S(X, Y)\varphi &= \frac{i}{2} d^\nabla W(X, Y) \cdot \varphi + \frac{1}{4}(W(X) \cdot W(Y) - W(Y) \cdot W(X)) \cdot \varphi, \\ \text{Ric}(X) \cdot \varphi &= -i \sum_{j=1}^n s_j \wedge d^\nabla W(X, s_j) \cdot \varphi - i \text{tr}_{\mathbf{g}} d^\nabla \Pi(X)(X)\varphi - \text{tr}_{\mathbf{g}}(W)W(X) \cdot \varphi + W^2(X) \cdot \varphi. \end{aligned}$$

*Proof.* The proof is a straightforward calculation in spin geometry, completely analogous to the one carried out for imaginary Killing spinors in [4] and for real W-Killing spinors in [15], for example.  $\square$

Finally we describe three classes of examples for complete Riemannian spin manifolds  $(\mathcal{M}, \mathbf{g})$  with imaginary W-Killing spinors.

**Example 6.1.** Let  $(\mathcal{M}, \mathbf{g})$  be compact and 2-dimensional. Since we are looking for compact 2-dimensional manifolds with a nowhere vanishing vector field (the Dirac current of the W-Killing spinor), it is enough to restrict ourself to the 2-torus  $T^2$  equipped with a metric  $\mathbf{g}$  conformally equivalent to the flat metric  $\mathbf{g}_0$ ,

$$\mathbf{g} := e^{2\sigma} \mathbf{g}_0 = e^{2\sigma} (a d\theta^2 + 2b d\theta d\rho + c d\rho^2), \quad a, c \in \mathbb{R}^+, b \in \mathbb{R}, ac > b^2.$$

Let  $U := f d\theta + h d\rho$  be a vector field on  $T^2$  and suppose that  $U$  has no zeros,  $a f^2 + 2b f h + c h^2 > 0$ . Then  $U$  is closed, respectively the endomorphism field

$$W := -\frac{1}{\|U\|_{\mathbf{g}}} \nabla^{\mathbf{g}} U$$

is  $\mathbf{g}$ -symmetric, if and only the functions  $\sigma, f, h \in C^\infty(T^2, \mathbb{R})$  satisfy the PDE

$$(b\partial_\theta - a\partial_\rho)(e^{2\sigma} f) = (-c\partial_\theta + b\partial_\rho)(e^{2\sigma} h).$$

We choose on  $(T^2, g)$  the trivial spin structure and consider the spinor field  $\varphi := \gamma \cdot v \in \Gamma(S) \simeq C^\infty(T^2, \Delta_2)$ , where  $\gamma \in C^\infty(T^2, \mathbb{C})$  is a function with  $|\gamma|^2 = \|U\|_{\mathbf{g}}$  and  $v \in \Delta_2$  is a fixed spinor with  $e_1 \cdot v = iv$  and  $\|v\| = 1$ . A direct calculation shows, that  $\varphi$  is an imaginary W-Killing spinor on  $(T^2, e^{2\sigma} g_0)$  with Dirac current  $U$ . Moreover, *all*  $\mathbf{g}$ -symmetric endomorphisms  $W$  and imaginary W-Killing spinors on the torus  $(T^2, e^{2\sigma} g_0)$  with trivial spin structure are of this form.

**Example 6.2.** Let  $(\mathcal{F}, \mathbf{g}_{\mathcal{F}})$  be a complete Riemannian spin manifold with a parallel spinor field, a Codazzi tensor  $T$  on  $(\mathcal{F}, \mathbf{g}_{\mathcal{F}})$  and  $b \in C^\infty(\mathbb{R}, \mathbb{R})$  a smooth function. Let  $A$  be the  $(1, 1)$ -tensor field on  $\mathcal{M} := \mathbb{R} \times \mathcal{F}$  given by

$$A = \begin{pmatrix} b(s) \cdot \text{Id}_{T\mathbb{R}} & 0 \\ 0 & e^s \left( T - \int_0^s b(r) e^{-r} dr \cdot \text{Id}_{T\mathcal{F}} \right) \end{pmatrix}.$$

Then  $\mathbf{g} := A^*(ds^2 + e^{-2s} \mathbf{g}_{\mathcal{F}})$  is a complete Riemannian metric on  $\mathcal{M} := \mathbb{R} \times \mathcal{F}$ ,  $W := A^{-1}$  is an invertible Codazzi tensor on  $(\mathcal{M}, \mathbf{g})$  and  $(\mathcal{M}, \mathbf{g})$  equipped with the spin structure induced by that of  $(\mathcal{F}, \mathbf{g}_{\mathcal{F}})$  admits an imaginary W-Killing spinor. On the other hand, all complete Riemannian spin manifolds  $(\mathcal{M}, \mathbf{g})$  with imaginary W-Killing spinor for an invertible Codazzi tensor  $W$  arise in this way. (For a proof see [6]).

**Example 6.3.** Let  $(\mathcal{M}, \mathbf{g})$  be a Riemannian manifold and  $W := b \text{Id}_{T\mathcal{M}}$ , where  $b$  is a smooth function on  $\mathcal{M}$  which is not identically zero. Then  $W$  is in general neither a Codazzi tensor nor invertible. Suppose, that  $U$  is a vector field and  $u$  a positive smooth function on  $\mathcal{M}$  such that  $\nabla U = -uW$  and  $\mathbf{g}(U, U) = u^2 > 0$ . Then,

$$\mathcal{L}_U \mathbf{g}(X, Y) = \mathbf{g}(\nabla_X U, Y) + \mathbf{g}(X, \nabla_Y U) = -2b\mathbf{g}(X, Y),$$

hence  $U$  is a conformal vector field on  $(\mathcal{M}, \mathbf{g})$  without zeros, which is closed as  $\nabla U$  is symmetric. Therefore  $(\mathcal{M}, \mathbf{g})$  is isometrically covered by the warped product  $(\mathbb{R} \times \mathcal{F}, ds^2 + h(s)^2 \mathbf{g}_{\mathcal{F}})$ , where  $(\mathcal{F}, \mathbf{g}_{\mathcal{F}})$  is a complete Riemannian manifold,  $h \in C^\infty(\mathbb{R}, \mathbb{R}^+)$  and  $b, U$  and  $u$  satisfy

$$\begin{aligned} b(\pi(s, x)) &= -(\ln h)'(s), \\ U(\pi(s, x)) &= d\pi(h(s) \partial_s(s, x)), \\ u(\pi(s, x)) &= h(s), \end{aligned}$$

where  $\pi$  denotes the covering map. If  $(\mathcal{M}, \mathbf{g})$  is in addition spin and  $U$  the Dirac current of an imaginary  $W = b \text{Id}_{T\mathcal{M}}$ -Killing spinor, then  $(\mathcal{F}, \mathbf{g}_{\mathcal{F}})$  is spin as well with a parallel spinor field. Conversely, any warped product  $\mathcal{M} := L \times_h \mathcal{F}$ , where  $(\mathcal{F}, \mathbf{g}_{\mathcal{F}})$  is a complete Riemannian manifold,  $L \in \{S^1, \mathbb{R}\}$  and  $h$  is a smooth positive function on  $L$  admits the closed conformal vector field  $U(s, x) = h(s)\partial_s(s, x)$  of length  $u = h$ , and the endomorphism  $W := b \text{Id}_{T\mathcal{M}}$  with  $b := -\ln(h)'$  satisfies  $\nabla U = -uW$ . If  $(\mathcal{F}, \mathbf{g}_{\mathcal{F}})$  is spin and has a parallel spinor field, then  $\mathcal{M} = L \times_h \mathcal{F}$  is spin as well with an imaginary  $W = b \text{Id}_{T\mathcal{M}}$ -Killing spinor. For a proof of all these statements see [16].

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